Needham, VDGF, Chapter 39, Exercise 10, p. 467.
G. Palmer, Aug 1, 2026, 10 pm, PT

10. Factorizes in $\mathbb{R}^4$*. Show that any 3-form in* $\mathbb{R}^4$ *can be expressed as the wedge product of three 1-forms.*

Use equation (35.3) to multiply three 1-forms $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, $\boldsymbol{\gamma}$. There is no restriction on the number of dimensions of the space in which the 1-forms are defined.

$$\begin{aligned}
\boldsymbol{\gamma} \wedge \boldsymbol{\varphi} \wedge \boldsymbol{\psi} &= \boldsymbol{\gamma} \wedge (-\boldsymbol{\psi} \wedge \boldsymbol{\varphi}) \\
&= \left[(-1)^{1\cdot 2} (-\boldsymbol{\psi} \wedge \boldsymbol{\varphi}) \wedge \boldsymbol{\gamma}\right] = \\
&= (\boldsymbol{\psi} \wedge \boldsymbol{\gamma} \wedge \boldsymbol{\varphi}) = (-\boldsymbol{\gamma} \wedge \boldsymbol{\psi} \wedge \boldsymbol{\varphi}) \\
&= \boldsymbol{\gamma} \wedge \boldsymbol{\varphi} \wedge \boldsymbol{\psi}
\end{aligned}$$

Since there are no restrictions on the 1-forms, once equation (35.3) is proven, it seems sufficient. But Needham doubtless wanted something more.

Let the following be general 1-forms in $\mathbb{R}^4$.

$$\begin{aligned}
\boldsymbol{\gamma} &= \gamma_1\, \boldsymbol{d}w + \gamma_2\, \boldsymbol{d}x + \gamma_3\, \boldsymbol{d}y + \gamma_4\, \boldsymbol{d}z \\
\boldsymbol{\varphi} &= \varphi_1\, \boldsymbol{d}w + \varphi_2\, \boldsymbol{d}x + \varphi_3\, \boldsymbol{d}y + \varphi_4\, \boldsymbol{d}z \\
,\quad \boldsymbol{\psi} &= \psi_1\, \boldsymbol{d}w + \psi_2\, \boldsymbol{d}x + \psi_3\, \mathrm{d}y + \psi_4\, \boldsymbol{d}z
\end{aligned}$$

Using indices 1..4 on the components γ_i, φ_j, ψ_k correspond to w, x, y, z, multiply out the three 1-forms as sums of wedge product 3-forms. Cancel wedge product summands such as $\boldsymbol{d}x \wedge \boldsymbol{d}z \wedge \boldsymbol{d}z$ containing bases where any two 1-forms are the same. Rearrange like combinations of three basis 1-forms and combine by summing the 1-form components. The result is:

$$\begin{aligned}
&(\gamma_0(\varphi_1\, \psi_2 - \varphi_2\, \psi_1) + \gamma_1(\varphi_2\, \psi_0 - \varphi_0\, \psi_2) + \gamma_2(\varphi_0\, \psi_1 - \varphi_1\, \psi_0)\, \boldsymbol{d}w \wedge \boldsymbol{d}x \wedge \boldsymbol{d}y \\
&(\gamma_0(\varphi_1\, \psi_3 - \varphi_3\, \psi_1) + \gamma_1(\varphi_3\, \psi_0 - \varphi_0\, \psi_3) + \gamma_3(\varphi_0\, \psi_1 - \varphi_1\, \psi_0)\, \boldsymbol{d}w \wedge \boldsymbol{d}x \wedge \boldsymbol{d}z \\
&(\gamma_0(\varphi_2\, \psi_3 - \varphi_3\, \psi_2) + \gamma_2(\varphi_3\, \psi_0 - \varphi_0\, \psi_3) + \gamma_3(\varphi_0\, \psi_2 - \varphi_2\, \psi_0)\, \boldsymbol{d}w \wedge \mathbf{dy} \wedge \boldsymbol{d}z \\
&(\gamma_1(\varphi_2\, \psi_3 - \varphi_3\, \psi_2) + \gamma_2(\varphi_3\, \psi_1 - \varphi_1\, \psi_3) + \gamma_3(\varphi_1\, \psi_2 - \varphi_2\, \psi_1)\, \mathbf{dx} \wedge \mathbf{dy} \wedge \boldsymbol{d}z
\end{aligned}$$

Since the 1-forms are general, these four 3-forms appear to represent all the possible combinations deriving from $\boldsymbol{\gamma}\boldsymbol{\cdot}(\boldsymbol{\varphi} \times \boldsymbol{\psi})$. They represent the volume of four parallelopipeds in 4-space.