Needham, VDGF, Chapter 39, Exercise 1, P. 465. September 7, 2025, 2:20 pm PDT

1. Dirac Delta Function as 1-Form.

(i) For any function g(x) in C[-1,1] define a corresponding 1-form $\tilde{g}(x)$ such that its action on the vector f(x) is given by

$$<\tilde{g}(x),\ f(x)> \ \equiv \int_{-1}^{1} g(x)\, f(x)\, dx.$$

Prove that $\tilde{g}$ is indeed a 1-form.

The proof will follow the proof for the Dirac bra given in §32.3.5, p. 352, which checks the definition (32.1) for linearity in multiplication and addition.

$$\tilde{g}(k_1 f_1(x) + k_2 f_2(x)) \equiv \int_{-1}^{1} g(x)\,(k_1 f_1(x) + k_2 f_2(x))\, dx$$

$$= k_1 \int_{-1}^{1} g(x) \left(f_1(x)\, dx + k_2 \int_{-1}^{1} f_2(x)\right) dx$$

Therefore $<\tilde{g}(x),\ f(x)> \ \equiv \int_{-1}^{1} g(x)\, f(x)\, dx$ is a 1-form.

(ii) Thus Dirac actually defined his delta function by insisting that

$$\int_{-1}^{1} \delta(x)\, f(x)\, dx = f(0) \qquad (39.1)$$

No ordinary function $\delta(x)$ can do this, but consider a smooth bell-shaped curve centred at 0, and imagine it growing higher and higher as it becomes narrower and narrower, all the while obeying (39.1), with $f(x) = 1$.

What is meant by “in the limit”? Does in mean in the limit as x approaches 0 from either side of the y axis? As x approaches 0 from both sides?

From Russell Herman, “The Dirac Delta Function” Libre Texts, Mathematics <https://math.libretexts.org-/Bookshelves/Differential_Equations/Introduction_to_Partial_Differential_Equations_(Herman)/09%3A_Transform_Techniques_in_Physics/9.04%3A_The_Dirac_Delta_Function> we have that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

and *Integration over a more general area gives*

$$\int_{a}^{b} \delta(x)\, dx = \begin{cases} 1, & 0 \in [a, b], \\ 0, & 0 \notin [a, b]. \end{cases}$$

Then if 0 is an element in [a,b], which it must be, since we are concerned with f(0),

$$\int_{-1}^{1} \delta(x)\, dx = 1.$$

Since f(x) = 1 for all x, f(0) = 1. Then

$$\int_{-1}^{1} \delta(x) f(x)\, dx = 1 \cdot \int_{-1}^{1} \delta(x)\, dx = 1 = f(0)$$

This recovers (39.1).