

Notes on Needham, Chapter 38

G. Palmer, October 19, 2026, 5 pm PT

I can't help thinking our text makes the topic of connection 1-forms a little more confusing than necessary, but that doesn't mean there is no added value in the presentation. However, I provide my own summary here, as I find it more readily apprehended and perhaps more useful.

I would center attention on the moving frame field represented with the matrix $[m]$.

$$[m] = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

$[m]$ changes attitude $[A]$ with respect to the fixed Euclidean frame field $[e]$ as it moves along a vector, curve, or surface.

$$[e] = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

It seems unorthodox to write the frame vectors of $[m]$ and $[e]$ as row vectors, but perhaps it is not a problem so long as one recognizes what they represent. For example,

$$m_1 = a_{11} e_1 + a_{12} e_2 + a_{13} e_3,$$

where a_{ij} stand for attitude components, and

$$m_j = \sum_i a_{ji} e_i$$

Then we can rewrite $[m]$ as the attitude matrix $[A]$ times the fixed frame field.

$$[m] = [A][e]$$

To show *change* in attitude, one takes external derivatives of the components of $[A]$ to obtain the “tipping” of each frame vector in the direction of another frame vector. That is, as m_i changes on a vector, one wants the projection of that change onto m_j for all i and j . The i 's and j 's index the same three frame vectors, but in various combinations. This is obtained elegantly with the external derivative applied to the inverse attitude matrix $[A]$ to yield a matrix of *connection 1-forms* $[\omega]$:

$$[\omega] = (d[A])[A]^T \quad (38.6)$$

This can be rewritten

$$\omega_{ij}(\mathbf{v}) = (\nabla_{\mathbf{v}} \mathbf{m}_i) \cdot \mathbf{m}_j = \sum_k (d a_{ik}) (\mathbf{v}) \mathbf{e}_k \cdot \mathbf{m}_j$$

The projection of the directional (external) derivative $\nabla_{\mathbf{v}} \mathbf{m}_i$ onto $\mathbf{m}_j$ is accomplished with the dot product.

Abstracting the vector away results in a bare connection 1-form:

$$\omega_{ij} = \sum_k (d a_{ik}) a_{jk} \quad (38.7)$$