

Notes on Needham, Chapter 37.

G. Palmer, October 7, 2026, 2:20 pm PT

P. 404. 37.1.1 Circulation and Work

On the first page of chapter 37 there are instructions to refer to chapter 27, specifically to (27.2), where the concept of circulation is introduced and defined as the circulation of a *vector field on a loop*.

$$C_L(V) = \oint_L (V) \cdot d\mathbf{r} = \oint_L V ds$$

But on p. 404, circulation $C_K(\varphi)$ is taken to be *fluid flow with velocity φ along a directed curve K* :

$$C_K \equiv \int_K \varphi \cdot d\mathbf{r}$$

This is a more specific model than that presented in chapter 27. The character φ shows that the model is being integrated with the algebra of forms. Here φ evidently represents the vector part of a 1-form. Accordingly, the labels and captions on [37.1] differ from those on [27.2][a], which otherwise appears to be the same diagram with the loop closed in [27.2][a] and open in [37.1]. In the opening paragraph of chapter 37, Needham also pointed out that one could alternatively think of φ as a force field in moving a particle along K , whereupon the integral represents work done in moving the particle.

It strikes me as a happy accident that one can use almost the same math for total fluid flow over a distance as for work done by a force field over a distance. It requires only that one conceive of force and velocity of fluid flow as vectors along a path through a vector field.

P. 420. Differentiation of (37.17).

This differentiation seemed a little obscure, so I worked through it with a little help from Vasco. Note that line 1 in the text corresponds to line 5 in this exercise.

$$\varphi = \frac{q}{2\pi r^2} [-y d\mathbf{x} + x d\mathbf{y}]$$

Note that $d\mathbf{r}^2 = 2 r d\mathbf{r}$ and $r^2 = x^2 + y^2$.

$$\left[\frac{2\pi}{q} \right] d\varphi = d\left[\frac{1}{r^2} \right] \wedge [-y d\mathbf{x} + x d\mathbf{y}] + \frac{1}{r^2} d[-y d\mathbf{x} + x d\mathbf{y}] \quad (\text{Leibniz})$$

$$= \partial_r \left(\frac{1}{r^2} \right) d\mathbf{r} \wedge [-y d\mathbf{x} + x d\mathbf{y}] + \frac{1}{r^2} [-1 \cdot d\mathbf{y} \wedge d\mathbf{x} + 1 \cdot d\mathbf{x} \wedge d\mathbf{y}]$$

$$= -\frac{2}{r^3} d\mathbf{r} \wedge [-y d\mathbf{x} + x d\mathbf{y}] + \frac{1}{r^2} [d\mathbf{x} \wedge d\mathbf{y} + d\mathbf{x} \wedge d\mathbf{y}]$$

$$\begin{aligned}
&= -\frac{2rdr}{r^4} \wedge [-ydx + xdy] + \frac{1}{r^2} 2 dx \wedge dy \\
&= -\frac{1}{r^4} dr^2 \wedge [-ydx + xdy] + \frac{1}{r^2} 2 dx \wedge dy \quad (\text{sub. } dr^2 \text{ for } 2rdr) \\
&= -\frac{1}{r^4} d(x^2 + y^2) \wedge [-ydx + xdy] + \frac{1}{r^2} 2 dx \wedge dy \quad (\text{sub. } x^2 + y^2 \text{ for } r^2) \\
&= -\frac{1}{r^4} 2(xdx + ydy) \wedge [-ydx + xdy] + \frac{1}{r^2} 2 dx \wedge dy \\
&= -\frac{1}{r^4} 2[xdx \wedge (-ydx) + xdx \wedge xdy + ydy \wedge (-ydx) + ydy \wedge xdy] + 2 \frac{1}{r^2} dx \wedge dy \\
&= -2 \frac{1}{r^4} [0 + xdx \wedge xdy - ydy \wedge ydx + 0] + 2 \frac{1}{r^2} dx \wedge dy \\
&= -2 \frac{1}{r^4} [x^2 dx \wedge dy + y^2 dx \wedge dy] + 2 \frac{1}{r^2} dx \wedge dy \\
&= 2 \left(-\frac{1}{r^4} [x^2 + y^2] + \frac{1}{r^2} \right) dx \wedge dy \\
&= 2 \left(-\frac{1}{r^4} r^2 + \frac{1}{r^2} \right) dx \wedge dy \quad (\text{sub. } r^2 \text{ for } x^2 + y^2) \\
&= 2 \left(-\frac{1}{r^2} + \frac{1}{r^2} \right) dx \wedge dy = 0
\end{aligned}$$

P. 420. 37.9.4. Geometrical Meaning of the Vortex 1-Form

Chapter 37 is concerned with flux and circulation: flux through boundaries and surfaces; and circulation round loops. It all seems somewhat intuitive and manageable until one arrives at 37.9 A Primer on de Rham Cohomology. Here we meet a 2-dimensional vortex field and learn that the 1-form φ of the vortex is closed (see previous), but it cannot be exact (it cannot itself be an external derivative of a function---a 0-form) because the circulation is non-zero. The latter is a reason I don't understand, although one can see intuitively from the diagram that the circulation is non-zero. And one can see that the origin is a singularity, so one cannot find a derivative at that point.

However, if we introduce a new simply connected region S into the vortex field, and a closed loop L into the region S , and we do not let S encircle the origin, then one can find derivatives of the vortex over the entire region S , and including L , and the circulation of the vortex on L will always sum to zero and have path independence, as in [37.12].

Next he introduces a dipole [37.13], which he names $\mathbf{X}$. It is unclear why a dipole can be represented as a 1-form. This seems a very different way to represent a dipole from that presented in *Visual Complex Analysis*. But regardless, he uses it like the vortex, to show that so long as we keep S outside the origin

and simply connected, X is exact in S . Here is what is curious. He introduces a loop L which completely encloses the origin and also passes through S , but so long as one confines calculations to the region S , X remains exact on L and path independent.

This is used to prove the Deformation Theorem for Closed 1-Forms. (p. 423).

P. 425-426. We leave it as an exercise to confirm by direct calculation that, indeed, $d\psi = 0$.

$$\psi = \frac{q}{r\pi^3} (x dx + y dy + z dz)$$

$$\frac{r\pi^3}{q} d\psi = d(x dx + y dy + z dz)$$

$$\begin{aligned} &= \partial_x x dx \wedge dx + \partial_y x dy \wedge dx + \partial_z x dz \wedge dx \\ &+ \partial_x y dx \wedge dy + \partial_y y dy \wedge dy + \partial_z y dz \wedge dy \\ &+ \partial_x z dx \wedge dz + \partial_y z dy \wedge dz + \partial_z z dz \wedge dz \end{aligned}$$

$$\begin{aligned} &= 1 \cdot 0 + 0 \cdot dy \wedge dx + 0 \cdot dz \wedge dx \\ &+ 0 \cdot dx \wedge dy + 1 \cdot 0 + 0 \cdot dz \wedge dy \\ &+ 0 \cdot dx \wedge dz + 0 \cdot dy \wedge dz + 1 \cdot 0 \\ &= 0 \implies d\psi = 0 \end{aligned}$$

As a lemma, first observe [exercise] that

$$dr = \frac{1}{2} (x dx + y dy + z dz).$$

$$r^2 = x^2 + y^2 + z^2$$

$$dr^2 = d(x^2 + y^2 + z^2)$$

$$2r dr = 2x dx + 2y dy + 2z dz$$

$$dr = \frac{1}{r} (x dx + y dy + z dz)$$

P. 426. Prove that Ψ is closed even though it is not exact. I.e. Show that $d\Psi = 0$ by showing that $\frac{4\pi}{q} d\Psi = 0$.

Let $(\text{flux} - 2) = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$.

$$d(\text{flux} - 2) = 3V \quad (\text{by calculation})$$

$$d\left(\frac{1}{r^3}\right) = \partial_r\left(\frac{1}{r^3}\right) dr = -\frac{3}{r^4} dr$$

$$dr = \frac{1}{r} (x dx + y dy + z dz) \quad (\text{p. 425})$$

$$dr \wedge (\mathbf{flux} - 2) = rV \quad (\text{by calculation})$$

$$\left[\frac{4\pi}{q}\right] d\Psi = d\left(\frac{1}{r^3}\right) \wedge (\mathbf{flux} - 2) + \frac{1}{r^3} d(\mathbf{flux} - 2) \quad (\text{Leibniz})$$

$$= -\frac{3}{r^4} dr \wedge (\mathbf{flux} - 2) + \frac{1}{r^3} 3V$$

$$= \frac{3}{r^3} \left(-\frac{1}{r} rV + V\right)$$

$$= (-1 + 1) V = 0$$