Needham, VDGF, Chapter 36, Notes
August 28, 2026, 5:30 pm PT

P. 393, Derivation of (36.3).
This derivation struck me as not completely obvious, so I present my version here.

$\mathbf{d}\boldsymbol{\varphi}(u, v) = \nabla_{\boldsymbol{u}}\, \boldsymbol{\varphi}(\boldsymbol{v}) - \nabla_{\boldsymbol{v}}\, \boldsymbol{\varphi}(\boldsymbol{u})$

$= u^i\, \partial_i\left(v^j\, \varphi_j\right) - v^i\, \partial_i\left(u^j\, \varphi_j\right)$

(because $\nabla_{\mathbf{u}} \to u^i\, \partial_i$, and $v \to v^j\, \varphi_j$)

$= u^i\, v^j\, \partial_i\, \varphi_j - u^j\, v^i\, \partial_i\, \varphi_j$

(by linearity of components u^i, u^j, v^i, v^j)

$= \partial_i\, \varphi_j \left\{u^i\, v^j - u^j\, v^i\right\}$
$= \partial_i\, \varphi_j \left\{\boldsymbol{d}x^i(\boldsymbol{u})\, \boldsymbol{d}x^j(\boldsymbol{v}) - \boldsymbol{d}x^j(\boldsymbol{u})\, \boldsymbol{d}x^i(\boldsymbol{v})\right\}$

(because $\boldsymbol{d}x^i(\boldsymbol{u}) = u^i$, pp. 356, 364 ln-6, p. 373, ¶3)
(ex.: $\boldsymbol{d}x^2(\boldsymbol{u}) = u^2$, $\boldsymbol{d}y(\boldsymbol{u}) = u^2$)

$= \partial_i\, \varphi_j\left(\boldsymbol{d}x^i \wedge \boldsymbol{d}x^j\right)(\boldsymbol{u}, \boldsymbol{v}) \Longrightarrow \mathrm{d}\varphi = \partial_i\, \varphi_j\left(\boldsymbol{d}x^i \wedge \boldsymbol{d}x^j\right)$ (36.3)

(by abstracting away from arbitrary vector fields **u** and **v**)

P. 394, §36.1. Show that $\mathbf{d}\boldsymbol{\varphi} = \boldsymbol{d}\left(\varphi_j\, \boldsymbol{d}x^j\right) = \boldsymbol{d}\varphi_j \wedge \boldsymbol{d}x^j$.

φ_j is a set of zero forms, each of which is comparable to f.

$\boldsymbol{d}f = \partial_i f\, \boldsymbol{d}x^i \quad \Longrightarrow \quad \mathbf{d}\boldsymbol{\varphi} = \partial_i\, \varphi_j\, \boldsymbol{d}x^i$ (p. 394, ln 1)

$\mathbf{d}\boldsymbol{\varphi} = \partial_i\, \varphi_j\left(\boldsymbol{d}x^i \wedge \boldsymbol{d}x^j\right) = \left(\partial_i\ \varphi_j\, \boldsymbol{d}x^i \wedge \boldsymbol{d}x^j\right) = \boldsymbol{d}\varphi_j \wedge \boldsymbol{d}x^j$

Or it can be derived with the Leibniz rule for forms (36.8) near the bottom of the same page.

$\mathbf{d}\boldsymbol{\varphi} = \boldsymbol{d}\left(\varphi_j\, \boldsymbol{d}x^j\right) = \boldsymbol{d}\varphi_j \wedge \boldsymbol{d}x^j + (-1)^0\, \varphi_j\, \boldsymbol{d}\left(\boldsymbol{d}x^j\right) = \boldsymbol{d}\varphi_j \wedge \boldsymbol{d}x^j$ (because $\boldsymbol{d}\left(\boldsymbol{d}x^j\right) = 0$)

P. 394, §36.2. *We find [**exercise**] that (36.4) generalizes in a way that is clearly part of a larger pattern:*

$$\mathbf{d\Psi} = \boldsymbol{d}\left(\Psi_{\mathrm{ij}} \wedge \boldsymbol{d}x^i \wedge \boldsymbol{d}x^j\right) = \boldsymbol{d}\Psi_{\mathrm{ij}} \wedge \boldsymbol{d}x^i \wedge \boldsymbol{d}x^j$$

Derivation:

$$\mathbf{d}\boldsymbol{\varphi} = \boldsymbol{d}\left(\varphi_j\, \mathrm{dx}^j\right) = \boldsymbol{d}\varphi_j \wedge \boldsymbol{d}x^j \qquad \text{From (36.4)}$$

$$\mathbf{d\Psi}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = \nabla_{\boldsymbol{v}_1} \mathbf{\Psi}(\boldsymbol{v}_2, \boldsymbol{v}_3) + \nabla_{\boldsymbol{v}_2} \mathbf{\Psi}(\boldsymbol{v}_3, \boldsymbol{v}_1) + \nabla_{\boldsymbol{v}_3} \mathbf{\Psi}(\boldsymbol{v}_1, \boldsymbol{v}_2) \qquad \text{From ¶ 2 of 36.2}$$

Then abstract away vector fields, as in (36.3) and apply (36.4) to the 2-form.

$$= \boldsymbol{d}\Psi_{23} \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 + \boldsymbol{d}\Psi_{31} \wedge \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 + \boldsymbol{d}\Psi_{12} \wedge \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2$$

$$= \boldsymbol{d}\Psi_{\mathrm{ij}} \wedge \boldsymbol{d}x^i \wedge \boldsymbol{d}x^j \qquad \text{By abstraction of indices.}$$

P. 394, §36.3.P1. *It is simple to verify [**exercise**] that if f is a function (0-form) then*

$$\boldsymbol{d}(f\mathbf{\Psi}) = \boldsymbol{d}f \wedge \mathbf{\Psi} \; + \; f\boldsymbol{d}\mathbf{\Psi} \qquad \textit{(36.7)}$$

The wedge product is not used for multiplication with 0-forms such as f. **d**f is a 1-form, so the wedge product can be used with other 1-forms and higher order forms. So this derivation is a simple application of the Leibniz rule for forms.

Suppose $f = 2\,x^2\,y$. Then $\boldsymbol{d}f(\boldsymbol{v}) = \left(4\,\mathrm{xy} + 2\,x^2\right)\boldsymbol{d}x^i(\mathbf{v})$ for some $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $\mathbf{d}\mathrm{f}(\mathbf{v}) = 4\,\mathrm{ab} \; + \; 2\,b^2$.

P. 395. 36.4.1. Prove that $\boldsymbol{d}^2 = 0$.

Let i = 1, 2,...p; j = 1, 2,...p; $\mathbf{d}\mathrm{x}^i \wedge \mathbf{d}\mathrm{x}^j = 0$, where $i = j$.

$$\begin{aligned}
\boldsymbol{d}^2\,\varphi = \boldsymbol{d}(\boldsymbol{d}\varphi) &= \boldsymbol{d}\left[\partial_i\,\varphi_j\left(\boldsymbol{d}x^i \wedge \boldsymbol{d}x^j\right)\right] \\
&= \boldsymbol{d}\left\{\partial_i\,\varphi_j\right\} \wedge \boldsymbol{d}x^i \wedge \boldsymbol{d}x^j \quad \text{[From Leibniz product rule for forms.]} \\
&= \left\{\left[\partial_k\,\partial_i\,\varphi_j\right]\boldsymbol{d}x^k \wedge \boldsymbol{d}x^i\right\} \wedge \boldsymbol{d}x^j \quad \text{[From df} = \partial_i\,\text{fdx}^i\text{, p. 394, ln 1. Sub } \partial_i\,\varphi_j \text{ for f and k for i.]} \\
&= \left\{\partial_1\,\partial_1\,\varphi_j\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^1 + \partial_1\,\partial_2\,\varphi_j\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 + \partial_2\,\partial_1\,\varphi_j\,\boldsymbol{d}x^2 \wedge \boldsymbol{d}x^1 \ldots + \partial_p\,\partial_p\,\varphi_j\,\boldsymbol{d}x^p \wedge \boldsymbol{d}x^p\right\} \wedge \boldsymbol{d}x^j \\
&= \left\{0 + \partial_1\,\partial_2\,\varphi_j\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 - \partial_2\,\partial_1\,\varphi_j\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 + \ldots + 0\right\} \wedge \boldsymbol{d}x^j \\
&= \{0 + 0 + \ldots + 0\} \wedge \boldsymbol{d}x^j = 0
\end{aligned}$$

The ki and ik wedge products cancel because $\partial_i\,\partial_k\,\varphi_j = \partial_k\,\partial_i\,\varphi_j$ and $\left(\boldsymbol{d}x^k \wedge \boldsymbol{d}x^i\right) = -\left(\boldsymbol{d}x^i \wedge \boldsymbol{d}x^k\right)$. The square bracket notation in line 3 appears simply to signal a second level of partition within the curly brackets, but that seems unnecessary. One might use it to signal that the order of indices on the partial derivatives changes as the 1-forms are multiplied out as wedge products. Alternatively, it may signal that the order does not have to change because the second derivatives are equal no matter the order of ki or ik. Or one could use it to group the lower indices in contrast to the upper indices.

P. 397. Prove each of the three following facts:

Box: *If* $\mathbf{\Upsilon}$ *and* $\mathbf{\Phi}$ *are closed, then* $\mathbf{\Upsilon} \wedge \mathbf{\Phi}$ *is closed, too.*

Show that $\mathbf{d}(\mathbf{\Upsilon} \wedge \mathbf{\Phi}) = 0$.

$\mathbf{d\Upsilon} = \mathbf{d\Phi} = 0$ because $\mathbf{\Upsilon}$ and $\mathbf{\Phi}$ are closed

$\mathbf{d}(\mathbf{\Upsilon} \wedge \mathbf{\Phi}) = \mathbf{d\Upsilon} \wedge \mathbf{\Phi} + (-1)^{\deg \mathbf{\Upsilon}}\, \mathbf{\Upsilon} \wedge \mathbf{d\Phi}$ by (36.8)

$= 0 \wedge \mathbf{\Phi} + (-1)^{\deg \mathbf{\Upsilon}}\, \mathbf{\Upsilon} \wedge 0 = 0$

Box: If $\mathbf{\Upsilon}$ *is closed, then* $\mathbf{\Upsilon} \wedge \mathbf{d\Phi}$ *is closed for all* $\mathbf{\Phi}$*.*

Show that $\mathbf{d}(\mathbf{\Upsilon} \wedge \mathbf{d\Phi}) = 0$ if $\mathbf{\Upsilon}$ *is closed*.

$\mathbf{d}(\mathbf{\Upsilon} \wedge \mathbf{d\Phi}) = \mathbf{d\Upsilon} \wedge \mathbf{d\Phi} + (-1)^{\deg \mathbf{\Upsilon}}\, \mathbf{\Upsilon} \wedge \mathbf{dd\Phi}$ *by (36.8)*

$= 0 \wedge \mathbf{d\Phi} + (-1)^{\deg \mathbf{\Upsilon}}\, \mathbf{\Upsilon} \wedge \mathbf{0} = \mathbf{0}$ because $\mathbf{dd\Phi} = 0$

Box: *If deg* $\mathbf{\Phi}$ *is even, then* $\mathbf{\Phi} \wedge \boldsymbol{d\Phi}$ *is closed*.

$\mathbf{d\Phi} \wedge \mathbf{d\Phi} = 0$ only if Φ is of odd degree.

Show that $\mathbf{d}(\mathbf{\Phi} \wedge \mathbf{d\Phi}) = 0$ if deg $\mathbf{\Phi}$ is even.

$\mathbf{d}(\mathbf{\Phi} \wedge \mathbf{d\Phi}) = \mathbf{d\Phi} \wedge \mathbf{d\Phi} + (-1)^{\deg \mathbf{\Phi}}\, \mathbf{\Phi} \wedge \mathbf{dd\Phi}$ *by (36.8)*

Since degree of $\mathbf{\Phi}$ is even, degree of $\mathbf{d\Phi}$ is odd. Therefore, $\mathbf{d\Phi} \wedge \mathbf{d\Phi} = 0$ by 35.7, and $\mathbf{dd\Phi} = 0$, so one can write

$\mathbf{d}(\mathbf{\Phi} \wedge \mathbf{d\Phi}) = 0 + (-1)^{\deg \mathbf{\Phi}}\, \mathbf{\Phi} \wedge 0 = 0$

So *If deg* $\mathbf{\Phi}$ *is even, then* $\mathbf{\Phi} \wedge \boldsymbol{d\Phi}$ *is closed*.

P. 397, line -8. Show that $\mathbf{d}(f\boldsymbol{d}z) = \left(i\,\partial_x f - \partial_y f\right)\mathcal{A}$.

$$\mathbf{d}(f\boldsymbol{d}z) = \boldsymbol{d}f \wedge \boldsymbol{d}z$$

$$= \mathbf{d}(f\boldsymbol{d}z) = \left(\partial_x f\,\boldsymbol{d}x + \partial_y f\,\boldsymbol{d}y\right) \wedge (\boldsymbol{d}x + i\boldsymbol{d}y)$$

$$= \partial_x f\,\boldsymbol{d}x \wedge \boldsymbol{d}x + \partial_x f\,\boldsymbol{d}x \wedge i\boldsymbol{d}y + \partial_y f\,\boldsymbol{d}y \wedge \boldsymbol{d}x + \partial_y f\,\boldsymbol{d}y \wedge i\boldsymbol{d}y$$

$$= i\,\partial_x f\,\boldsymbol{d}x \wedge \boldsymbol{d}y - \partial_y f\,\boldsymbol{d}x \wedge \boldsymbol{d}y$$

$$= \left(i\,\partial_x f - \partial_y f\right)\boldsymbol{d}x \wedge \boldsymbol{d}y = \left(i\,\partial_x f - \partial_y f\right)\mathcal{A}$$

which is what we wanted.

P. 398, [**exercise**] (36.14).

$$\boldsymbol{d}(\overline{z}\,\boldsymbol{d}z)\,\overline{z} = \boldsymbol{d}\,\overline{z} \wedge \boldsymbol{d}z = \boldsymbol{d}(x - \mathrm{iy}) \wedge (\boldsymbol{d}x + i\boldsymbol{d}y) = (\boldsymbol{d}x - i\boldsymbol{d}y) \wedge (\boldsymbol{d}x + i\boldsymbol{d}y)$$

$$= \boldsymbol{d}x \wedge \boldsymbol{d}x + \boldsymbol{d}x \wedge i\boldsymbol{d}y - i\boldsymbol{d}y \wedge \boldsymbol{d}x - i \cdot i\boldsymbol{d}y \wedge \boldsymbol{d}y$$

$$= 0 + i\boldsymbol{d}x \wedge \boldsymbol{d}y + i\boldsymbol{d}x \wedge \boldsymbol{d}y - 0 = 2\,i\boldsymbol{d}x \wedge \boldsymbol{d}y = 2\,i\mathcal{A}$$

P. 400, ¶4, *leaving the simple proof [of (36.17) to you.*

$$\boldsymbol{\varphi} \wedge \boldsymbol{\Psi} = (\underline{\boldsymbol{\varphi}} \bullet \underline{\boldsymbol{\Psi}})\,\boldsymbol{V} \qquad (36.17)$$

Let $\underline{\boldsymbol{\varphi}} = \varphi_i = \begin{pmatrix} \varphi_1 \\ \varphi_3 \\ \varphi_3 \end{pmatrix}$

Let $\underline{\boldsymbol{\Psi}} = \Psi_i = \begin{pmatrix} \Psi_1 \\ \Psi_3 \\ \Psi_3 \end{pmatrix}$

Let $A^1 = \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3$, $A^2 = \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1$, $A^3 = \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2$ (see p. 399)

$$\boldsymbol{\varphi} \wedge \boldsymbol{\Psi} = \left(\varphi_1\,\boldsymbol{d}x^1 + \varphi_2\,\boldsymbol{d}x^2 + \varphi_3\,\boldsymbol{d}x^3\right) \wedge \left(\Psi_1\,A^1 + \Psi_2\,A^2 + \Psi_3\,A^3\right)$$

$$= \varphi_1\,\boldsymbol{d}x^1 \wedge \left(\Psi_1\,A^1 + \Psi_2\,A^2 + \Psi_3\,A^3\right) + \varphi_2\,\boldsymbol{d}x^2 \wedge \left(\Psi_1\,A^1 + \Psi_2\,A^2 + \Psi_3\,A^3\right) + \varphi_3\,\boldsymbol{d}x^3 \wedge \left(\Psi_1\,A^1 + \Psi_2\,A^2 + \Psi_3\,A^3\right)$$

$$= \left(\varphi_1\,\Psi_1\,\boldsymbol{d}x^1 \wedge A^1 + \varphi_1\,\Psi_2\,\boldsymbol{d}x^1 \wedge A^2 + \varphi_1\,\boldsymbol{d}x^1 \wedge \Psi_3\,A^3\right) + \ldots + \ldots$$

$$= \left(\varphi_1\,\Psi_1\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 + \varphi_1\,\Psi_2\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 + \varphi_1\,\boldsymbol{d}x^1 \wedge \Psi_3\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2\right) + \ldots + \ldots$$

$$= \left(\varphi_1\,\Psi_1\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 + 0 + 0\right) + \left(\varphi_2\,\Psi_2\,\boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 + 0 + 0\right) + \left(\varphi_3\,\Psi_3\,\boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 + 0 + 0\right)$$
[$\boldsymbol{d}x^i \wedge \boldsymbol{d}x^i = 0$ by (35.4)]

$$= \varphi_1\,\Psi_1\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 + \varphi_2\,\Psi_2\,\boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 + \varphi_3\,\Psi_3\,\boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2$$

$= (\varphi_1\,\Psi_1 + \varphi_2\,\Psi_2 + \varphi_3\,\Psi_3)\,\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3$ (shift 2nd and 3rd basis terms by two positions to obtain $+\,\mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3$)

$$= (\underline{\boldsymbol{\varphi}} \bullet \underline{\boldsymbol{\Psi}})\,\boldsymbol{V}$$

P. 400, ¶5, [**exercise**] Derive $\nabla \bullet [f\,\underline{\boldsymbol{\Psi}}] = (\nabla f) \bullet \underline{\boldsymbol{\Psi}} + f\,\nabla \bullet \underline{\boldsymbol{\Psi}}$ directly.

I'm not sure what is meant by "directly", but the following seems to work well.

$$\nabla \bullet [f\,\underline{\boldsymbol{\Psi}}] = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \bullet \begin{pmatrix} \mathrm{f}\Psi_1 \\ f\,\Psi_2 \\ \mathrm{f}\Psi_3 \end{pmatrix} = \begin{pmatrix} \partial_1 (f\,\Psi_1) \\ \partial_2 (f\,\Psi_2) \\ \partial_3 (\mathrm{f}\Psi_3) \end{pmatrix} = \begin{pmatrix} \partial_1 f\,\Psi_1 \\ \partial_2 f\,\Psi_2 \\ \partial_3 \mathrm{f}\Psi_3 \end{pmatrix} + \begin{pmatrix} f\,\partial_1 \Psi_1 \\ f\,\partial_2 \Psi_2 \\ f\,\partial_3 \Psi_3 \end{pmatrix}$$

$$= \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix} \bullet \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} + f \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \bullet \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = (\nabla f) \bullet \underline{\boldsymbol{\Psi}} + f\,\nabla \bullet \underline{\boldsymbol{\Psi}} = ((\nabla f) + f\,\nabla) \bullet \underline{\boldsymbol{\Psi}}$$

[The last part may not be valid notation after the final equal sign, but it does give a more vivid impression of the contributions of the gradients of f and $\underline{\boldsymbol{\Psi}}$.]

Now derive $\nabla \times [f\,\underline{\boldsymbol{\Psi}}] = (\nabla f) \times \underline{\boldsymbol{\Psi}} + f\,\nabla \times \underline{\boldsymbol{\Psi}}$ directly.

$$\nabla \times [f\,\underline{\boldsymbol{\Psi}}] = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} \mathrm{f}\Psi_1 \\ f\,\Psi_2 \\ \mathrm{f}\Psi_3 \end{pmatrix} = \begin{pmatrix} \partial_2 (f\,\Psi_3) - \partial_3 (f\,\Psi_2) \\ \partial_3 (f\,\Psi_1) - \partial_1 (f\,\Psi_3) \\ \partial_1 (f\,\Psi_2) - \partial_2 (f\,\Psi_1) \end{pmatrix}$$

Apply Leibnitz differentiation to every matrix entry and rearrange.

$$= \begin{pmatrix} \partial_2 (f)\, \Psi_3 + f\, \partial_2\, \Psi_3 - \partial_3 (f)\, \Psi_2 - f\, \partial_3\, \Psi_2 \\ \partial_3 (f)\, \Psi_1 + f\, \partial_3\, \Psi_1 - \partial_1 (f)\, \Psi_3 - f\, \partial_1\, \Psi_3 \\ \partial_1 (f)\, \Psi_2 + f\, \partial_1\, \Psi_2 - \partial_2 (f)\, \Psi_1 - f\, \partial_2\, \Psi_1 \end{pmatrix} = \begin{pmatrix} (\partial_2 (f)\, \Psi_3 - \partial_3 (f)\, \Psi_2) + (f\, \partial_2\, \Psi_3 - f\, \partial_3\, \Psi_2) \\ (\partial_3 (f)\, \Psi_1 - \partial_1 (f)\, \Psi_3) + (f\, \partial_3\, \Psi_1 - f\, \partial_1\, \Psi_3) \\ (\partial_1 (f)\, \Psi_2 - \partial_2 (f)\, \Psi_1) + (f\, \partial_1\, \Psi_2 - f\, \partial_2\, \Psi_1) \end{pmatrix}$$

$$= \begin{pmatrix} \partial_2 (f)\, \Psi_3 - \partial_3 (f)\, \Psi_2 \\ \partial_3 (f)\, \Psi_1 - \partial_1 (f)\, \Psi_3 \\ \partial_1 (f)\, \Psi_2 - \partial_2 (f)\, \Psi_1 \end{pmatrix} + f \begin{pmatrix} \partial_2\, \Psi_3 - \partial_3\, \Psi_2 \\ \partial_3\, \Psi_1 - \partial_1\, \Psi_3 \\ \partial_1\, \Psi_2 - \partial_2\, \Psi_1 \end{pmatrix}$$

$$= \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix} \times \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} + f \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}$$

$$= (\nabla f) \times \underline{\mathbf{\Psi}} + f\, \nabla \times \underline{\mathbf{\Psi}} = (\nabla f + f\, \nabla) \times \underline{\mathbf{\Psi}}$$

[Same comment regarding validity of notation after final equal sign.]

P. 401. Discussion/review of page and the idea that we can use the above ideas to encapsulate Maxwell's complete laws of electromagnetism in just two equations.

The two **source-free** equations:

$\nabla \bullet \underline{\boldsymbol{B}} = 0$ Gradient of the coefficient functions $\underline{\boldsymbol{B}}$ of the 2-form magnetic field **B** equals 0.

$\nabla \times \underline{\boldsymbol{E}} + \partial_t \underline{\boldsymbol{B}} = 0$ Curl of coefficient functions $\underline{\boldsymbol{E}}$ of electrical field **E** plus partial wrt time t of coefficient functions $\underline{\boldsymbol{B}}$ of 2-form magnetic field **B** equals 0.

Needham presents the general differential **d**f of any function in four space-time dimensions in terms of $\boldsymbol{d_S}$ (the spatial part) plus a differential on the time part t.

$\boldsymbol{d}f = \boldsymbol{d_S}\, f + \partial_t f\, \boldsymbol{d}t$

Then he says "Now consider the exterior derivative of the Faraday 2-form, (34.22):

$\mathbf{dF} = \boldsymbol{d}(\epsilon \wedge \boldsymbol{d}t) + \mathbf{dB}$" [$\epsilon$ should be the electric field 1-form $\epsilon = E_x\, \boldsymbol{d}x + E_y\, \boldsymbol{d}y + E_z\, \boldsymbol{d}z$ (34.19), if Needham has maintained that identity.]

The two summands of the Faraday 2-form are to be evaluated separately, and in turn, starting with $\boldsymbol{d_S}(\epsilon) \wedge \boldsymbol{d}t$. The result should be the "[flux 2-form of $\nabla \times \underline{\boldsymbol{E}}$] $\wedge$ dt", as shown in (36.5) and the equation in ¶4 of §**36.5**, with the substitution of $\underline{\boldsymbol{E}}$ for $\boldsymbol{\varphi}$.

$$\boldsymbol{d}(\epsilon \wedge \boldsymbol{d}t) = \boldsymbol{d_S}(\epsilon) \wedge \boldsymbol{d}t = \left(\partial_i E_j\left(\boldsymbol{d}x^i \wedge \boldsymbol{d}x^j\right)\right) \wedge \boldsymbol{d}t \quad \text{(by 36.4 and 36.5)}$$

One could also write this more iconically as

$$[(\nabla \times \underline{\boldsymbol{E}})_i \bullet \mathrm{A}_i] \wedge \boldsymbol{d}t,$$

where $(\nabla \times \underline{\boldsymbol{E}})_1 \bullet \mathrm{A}_1 = (\partial_2 E_3 - \partial_3 E_2)\, \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3$. Needham writes it as the "[flux 2-form of $\nabla \times \underline{\boldsymbol{E}}$] $\wedge$ dt".

The result is a 3-form. ?[Since **dF** is a sum of forms and itself a form, its second term **dB** must also be a 3-form.]

Needham reduces $\boldsymbol{d_S}\,\boldsymbol{B}$ to $(\nabla \bullet \underline{\boldsymbol{B}})\,\boldsymbol{V}$. Then, to complete the general differential $\boldsymbol{d}f = \boldsymbol{d_S}\,f + \partial_t f\, \boldsymbol{d}t$, all that remains is to find $\partial_t \boldsymbol{B}\, \boldsymbol{d}t = \partial_t(\boldsymbol{B}) \wedge \boldsymbol{d}t$. By (34.20), **B** is the flux 2-form

$$\boldsymbol{B} = B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + B_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + B_z(\boldsymbol{d}x \wedge \boldsymbol{d}y).$$

Taking just the first term,

$$\partial_t [B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z))] \wedge \boldsymbol{d}t = \partial_t B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) \wedge \boldsymbol{d}t$$

When all three terms are written, the result is written $\partial_t \underline{\boldsymbol{B}} \wedge \boldsymbol{d}t$. Needham points out that $\partial_t \underline{\boldsymbol{B}}$ has the flux 2-form $\partial_t \boldsymbol{B}$. I don't understand exactly what that is in terms of vectors. If one grants that $\mathbf{dB} = (\nabla \bullet \underline{\boldsymbol{B}})\,\boldsymbol{V} + \partial_t \underline{\boldsymbol{B}} \wedge \boldsymbol{d}t$, then one can write (in the order created),

$$\mathbf{dF} = [(\nabla \times \underline{\boldsymbol{E}})_i \bullet \mathrm{A}_i] \wedge \boldsymbol{d}t + (\nabla \bullet \underline{\boldsymbol{B}})\,\boldsymbol{V} + \partial_t \underline{\boldsymbol{B}} \wedge \boldsymbol{d}t$$

then rearrange and put the two terms under wedge **d**t in curly brackets

$$= (\nabla \bullet \underline{\boldsymbol{B}})\,\boldsymbol{V} + \{(\nabla \times \underline{\boldsymbol{E}})_i \bullet \mathrm{A}_i + \partial_t \underline{\boldsymbol{B}}\} \wedge \boldsymbol{d}t$$

This format appears to have the advantage of adding two time-dependent flux terms together before adding them to a strictly spatial divergence term. A very concrete space-time example would be useful at this point.

P. 402. In 4-space, these may be useful points of reference.

$$\star \boldsymbol{J} = -\rho \boldsymbol{V} + \{\text{flux 2} - \text{form of } \boldsymbol{j}\} \wedge \boldsymbol{d}t \qquad (36.18)$$

$$\boldsymbol{d} \star \boldsymbol{F} = 4\,\pi \star \boldsymbol{J} = 4\,\pi[-\rho \boldsymbol{V} + \{\text{flux 2} - \text{form of } \boldsymbol{j}\} \wedge \boldsymbol{d}t] \quad \text{(p. 402)}$$

$$= 4\,\pi\left[-\rho\,\boldsymbol{d}x \wedge \boldsymbol{d}y \wedge \boldsymbol{d}z + \left\{j^1\,\boldsymbol{d}y \wedge \boldsymbol{d}z + j^2\,\boldsymbol{d}z \wedge \boldsymbol{d}x + j^3\,\boldsymbol{d}x \wedge \boldsymbol{d}y\right\} \wedge \boldsymbol{d}t\right]$$