Needham, VDGF, Chapter 35, Notes
November 25, 2025

§35.2. *Assuming (35.3) to be true, we also deduce that if the degree of* $\boldsymbol{\Omega}$ *is odd, then [**exercise**],*

$$\boldsymbol{\Omega} \wedge \boldsymbol{\Omega} = 0.$$

$$\boldsymbol{\Psi} \wedge \boldsymbol{\Omega} = (-1)^{pq}\, \boldsymbol{\Omega} \wedge \boldsymbol{\Psi}$$
$$\boldsymbol{\Omega} \wedge \boldsymbol{\Omega} = (-1)^{q^2}\, \boldsymbol{\Omega} \wedge \boldsymbol{\Omega}$$

If q is odd, then q·q is also odd and

$$\boldsymbol{\Omega} \wedge \boldsymbol{\Omega} = -\boldsymbol{\Omega} \wedge \boldsymbol{\Omega}$$

Therefore, $\boldsymbol{\Omega} \wedge \boldsymbol{\Omega} = 0$.

P. 390. ¶3. *To achieve our goal, we must try to write down an expression that is antisymmetric under exchange of any pair of 1-forms. It is possible to do this [exercise] by working backwards from (35.7), but is is actually simpler to attack the problem head on.*

To attack the problem head-on, use (35.1):

$$(\boldsymbol{\Psi} \wedge \boldsymbol{\sigma})\,(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = \boldsymbol{\Psi}(\boldsymbol{v}_1, \boldsymbol{v}_2)\, \boldsymbol{\sigma}(\boldsymbol{v}_3) + \boldsymbol{\Psi}(\boldsymbol{v}_3, \boldsymbol{v}_1)\, \boldsymbol{\sigma}(\boldsymbol{v}_2) + \boldsymbol{\Psi}(\boldsymbol{v}_2, \boldsymbol{v}_3)\, \boldsymbol{\sigma}(\boldsymbol{v}_1)$$

Our solution begins by rewriting $(\boldsymbol{\Psi} \wedge \boldsymbol{\sigma})\,(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) \equiv \left(\underset{1}{\sigma} \wedge \underset{2}{\sigma}\right) \wedge \underset{3}{\sigma}$. We assume that switching any two adjacent $\underset{i,j,k}{\sigma}$ has the same effect on symmetry as switching any two vectors, so we work with just the sigmas and rewrite (35.1).

$$\left(\underset{1}{\sigma} \wedge \underset{2}{\sigma}\right) \wedge \underset{3}{\sigma} = \left(\underset{1}{\sigma} \wedge \underset{2}{\sigma}\right) \underset{3}{\sigma} + \left(\underset{3}{\sigma} \wedge \underset{1}{\sigma}\right) \underset{2}{\sigma} + \left(\underset{2}{\sigma} \wedge \underset{3}{\sigma}\right) \underset{1}{\sigma}$$

$$\left(\underset{1}{\sigma} \wedge \underset{2}{\sigma}\right) \underset{3}{\sigma} = \left(\underset{1}{\sigma} \otimes \underset{2}{\sigma} - \underset{2}{\sigma} \otimes \underset{1}{\sigma}\right) \underset{3}{\sigma} = \underset{1}{\sigma} \otimes \underset{2}{\sigma} \otimes \underset{3}{\sigma} - \underset{2}{\sigma} \otimes \underset{1}{\sigma} \otimes \underset{3}{\sigma} \qquad \text{(a)}$$

$$\left(\underset{3}{\sigma} \wedge \underset{1}{\sigma}\right) \underset{2}{\sigma} = \left(\underset{3}{\sigma} \otimes \underset{1}{\sigma} - \underset{1}{\sigma} \otimes \underset{3}{\sigma}\right) \underset{2}{\sigma} = \underset{3}{\sigma} \otimes \underset{1}{\sigma} \otimes \underset{2}{\sigma} - \underset{1}{\sigma} \otimes \underset{3}{\sigma} \otimes \underset{2}{\sigma} \qquad \text{(b)}$$

$$\left(\underset{2}{\sigma} \wedge \underset{3}{\sigma}\right) \underset{1}{\sigma} = \left(\underset{2}{\sigma} \otimes \underset{3}{\sigma} - \underset{3}{\sigma} \otimes \underset{2}{\sigma}\right) \underset{1}{\sigma} = \underset{2}{\sigma} \otimes \underset{3}{\sigma} \otimes \underset{1}{\sigma} - \underset{3}{\sigma} \otimes \underset{2}{\sigma} \otimes \underset{1}{\sigma} \qquad \text{(c)}$$

Each of (a), (b), and (c) is antisymmetric under a swap of the first two 1-forms, so their sum must be antisymmetric. Sum (a), (b), and (c), switching the order of (b) and (c) and and collect the positive terms in line 1 and the negative terms in line 2 to obtain (35.8).

$$\underset{1}{\sigma}\wedge\underset{2}{\sigma}\wedge\underset{3}{\sigma} = \underset{1}{\sigma}\otimes\underset{2}{\sigma}\otimes\underset{3}{\sigma} + \underset{2}{\sigma}\otimes\underset{3}{\sigma}\otimes\underset{1}{\sigma} + \underset{3}{\sigma}\otimes\underset{1}{\sigma}\otimes\underset{2}{\sigma}$$

$$-\underset{2}{\sigma}\otimes\underset{1}{\sigma}\otimes\underset{3}{\sigma} - \underset{3}{\sigma}\otimes\underset{2}{\sigma}\otimes\underset{1}{\sigma} - \underset{1}{\sigma}\otimes\underset{3}{\sigma}\otimes\underset{2}{\sigma}$$

This is what we wanted. What about the replacement of multiplication signified by juxtaposition, which is rewritten with the symbol for tensor multiplication, as in $\left(\underset{1}{\sigma}\otimes\underset{2}{\sigma} - \underset{2}{\sigma}\otimes\underset{1}{\sigma}\right)\underset{3}{\sigma} = \underset{1}{\sigma}\otimes\underset{2}{\sigma}\otimes\underset{3}{\sigma} - \underset{2}{\sigma}\otimes\underset{1}{\sigma}\otimes\underset{3}{\sigma}$? Since the tensor difference in parentheses is multiplied by $\underset{3}{\sigma}$, also a tensor, one can deduce it is correct to use the symbol for tensor multiplication: $\otimes$.

P. 191, Ln -2. *If $\mathcal{V}$ denotes the volume 4-form of spacetime, then [**exercise**],*

$$\mathbf{\Psi}\wedge\mathbf{\Psi} = 2\,\boldsymbol{d}t\wedge\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z = 2\mathcal{V}$$

$$\mathbf{\Psi} = \boldsymbol{d}t\wedge\boldsymbol{d}x + \boldsymbol{d}y\wedge\boldsymbol{d}z$$

$$\mathbf{\Psi}\wedge\mathbf{\Psi} = (\boldsymbol{d}t\wedge\boldsymbol{d}x + \boldsymbol{d}y\wedge\boldsymbol{d}z)\wedge(\boldsymbol{d}t\wedge\boldsymbol{d}x + \boldsymbol{d}y\wedge\boldsymbol{d}z)$$

$$= (\boldsymbol{d}t\wedge\boldsymbol{d}x)\wedge(\boldsymbol{d}t\wedge\boldsymbol{d}x) + (\boldsymbol{d}t\wedge\boldsymbol{d}x)\wedge(\boldsymbol{d}y\wedge\boldsymbol{d}z) + (\boldsymbol{d}y\wedge\boldsymbol{d}z)\wedge(\boldsymbol{d}t\wedge\boldsymbol{d}x) + (\boldsymbol{d}y\wedge\boldsymbol{d}z)\wedge(\boldsymbol{d}y\wedge\boldsymbol{d}z)$$

Move **d**t one place to left in first summand and move **d**y one place to left in second summand to obtain zeros with wedge product of equal 1-forms.

$$= -\boldsymbol{d}t\wedge\boldsymbol{d}t\wedge\boldsymbol{d}x\wedge\boldsymbol{d}x + \boldsymbol{d}t\wedge\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z + \boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t\wedge\boldsymbol{d}x - \boldsymbol{d}y\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}z$$

$$= -0\wedge0 + \boldsymbol{d}t\wedge\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z + \boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t\wedge\boldsymbol{d}x - 0\wedge0$$

$$= \boldsymbol{d}t\wedge\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z + \boldsymbol{d}y\wedge\boldsymbol{d}z\wedge\boldsymbol{d}t\wedge\boldsymbol{d}x$$

In the 2nd wedge product, move **d**t two places left and then move **d**x two places left. The even sum of four moves (swaps) conserves sign.

$$= \boldsymbol{d}t\wedge\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z + \boldsymbol{d}t\wedge\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z$$

$$\mathbf{\Psi}\wedge\mathbf{\Psi} = 2\,\boldsymbol{d}t\wedge\boldsymbol{d}x\wedge\boldsymbol{d}y\wedge\boldsymbol{d}z = 2\mathcal{V}$$