Needham, VDGF, Chapter 34, Notes
July 3, 2026, 3:30 pm

§34.3. [exercise]. Split product of two 1-forms into sum of symmetric tensor and an antisymmetric tensor.

then by swapping the 1-forms in the tensor products (instead of swapping the vectors) we find that

$$\boldsymbol{\varphi} \otimes \boldsymbol{\psi} = \tfrac{1}{2}[\boldsymbol{\varphi} \otimes \boldsymbol{\psi} + \boldsymbol{\psi} \otimes \boldsymbol{\varphi}] + \tfrac{1}{2}[\boldsymbol{\varphi} \otimes \boldsymbol{\psi} - \boldsymbol{\psi} \otimes \boldsymbol{\varphi}]$$

See p. 369 for the framework used here. Since the vectors remain in fixed order, we can abstract them away immediately. On p. 369 paragraph 2, Needham wrote *Now let us try to do something analogous with a general tensor* $\boldsymbol{E}(\boldsymbol{v}, \boldsymbol{w})$ *of valence* $\{0,\ 2\}^T$. Here, we try to do something analogous by swapping the 1-forms in the tensor products:

Let $\boldsymbol{\varphi \otimes \psi = E(\varphi,\ \psi)} \equiv f(x)$
and let $\boldsymbol{\psi \otimes \varphi = E(\psi,\ \varphi)} \equiv f(-x)$

By analogy, the symmetric and asymmetric tensors are:

$$\boldsymbol{E^+(\varphi,\ \psi) \equiv \frac{\varphi \otimes \psi + \psi \otimes \varphi}{2}}$$

$$\boldsymbol{E^-(\varphi,\ \psi) \equiv \frac{\varphi \otimes \psi - \psi \otimes \varphi}{2}}$$

$$\boldsymbol{E(\varphi,\ \psi) \equiv E^+(\varphi,\ \psi) + E^-(\varphi,\ \psi) \equiv} \frac{\boldsymbol{\varphi \otimes \psi + \psi \otimes \varphi}}{2} + \frac{\boldsymbol{\varphi \otimes \psi - \psi \otimes \varphi}}{2}$$

This is what we wanted.

In the last equation, Needham shows that

$$(\boldsymbol{\varphi} \wedge \boldsymbol{\psi})(\boldsymbol{v}_1,\ \boldsymbol{v}_2) = -(\boldsymbol{\psi} \wedge \boldsymbol{\varphi})(\boldsymbol{v}_1,\ \boldsymbol{v}_2)$$

This seems an opportunity missed to show that swapping both the 1-forms and the vectors gives an equality without change of sign:

$$(\boldsymbol{\varphi} \wedge \boldsymbol{\psi})(\boldsymbol{v}_1,\ \boldsymbol{v}_2) = -(\boldsymbol{\psi} \wedge \boldsymbol{\varphi})(\boldsymbol{v}_1,\ \boldsymbol{v}_2) = (\boldsymbol{\psi} \wedge \boldsymbol{\varphi})(\boldsymbol{v}_2,\ \boldsymbol{v}_1)$$

§34.5, p. 376. *If we now go up one dimension to* $\mathbb{R}^3$*, then [**exercise**], exactly the same reasoning yields*

$$\boldsymbol{\Psi} = \Psi_{23}(\boldsymbol{d}x \wedge \boldsymbol{d}y) + \Psi_{31}(\boldsymbol{d}x \wedge \boldsymbol{d}z) + \Psi_{12}(\boldsymbol{d}y \wedge \boldsymbol{d}z)$$

confirming (34.7) for this case.

If $\mathbf{\Psi}$ were merely a generic tensor of valence $\left\{ \begin{matrix} 0 \\ 2 \end{matrix} \right\}$ in $\mathbb{R}^3$, then we know from (33.2) that it can be expanded as

$$\begin{aligned}\mathbf{\Psi} = {} & \Psi_{11}(\boldsymbol{d}x \otimes \boldsymbol{d}x) + \Psi_{12}(\boldsymbol{d}x \otimes \boldsymbol{d}y) + \Psi_{21}(\boldsymbol{d}y \otimes \boldsymbol{d}x) + \Psi_{22}(\boldsymbol{d}y \otimes \boldsymbol{d}y) + \\ & \Psi_{13}(\boldsymbol{d}x \otimes \boldsymbol{d}z) + \Psi_{31}(\boldsymbol{d}z \otimes \boldsymbol{d}x) + \Psi_{23}(\boldsymbol{d}y \otimes \boldsymbol{d}z) + \Psi_{32}(\boldsymbol{d}z \otimes \boldsymbol{d}y) + \Psi_{33}(\boldsymbol{d}z \otimes \mathrm{dz})\end{aligned}$$

But, because $\mathbf{\Psi}$ is a 2-form, $\Psi_{11} = \Psi_{22} = \Psi_{33} = 0$, and

$$\begin{aligned}\Psi_{12} &= -\Psi_{21} \\ \Psi_{31} &= -\Psi_{13} \\ \Psi_{23} &= -\Psi_{32}.\end{aligned}$$

Therefore,

$$\mathbf{\Psi} = \Psi_{12}(\boldsymbol{d}x \otimes \boldsymbol{d}y) + \Psi_{21}(\boldsymbol{d}y \otimes \boldsymbol{d}x) + \Psi_{13}(\boldsymbol{d}x \otimes \boldsymbol{d}z) + \Psi_{31}(\boldsymbol{d}z \otimes \boldsymbol{d}x) + \Psi_{23}(\boldsymbol{d}y \otimes \boldsymbol{d}z) + \Psi_{32}(\boldsymbol{d}z \otimes \boldsymbol{d}y)$$

But $\Psi_{12}(\boldsymbol{d}x \otimes \boldsymbol{d}y) = -\Psi_{21}(\boldsymbol{d}y \otimes \boldsymbol{d}x)$, so

$$\Psi_{12}(\boldsymbol{d}x \otimes \boldsymbol{d}y) + \Psi_{21}(\boldsymbol{d}y \otimes \boldsymbol{d}x) = \Psi_{12}(\boldsymbol{d}x \otimes \boldsymbol{d}y) - \Psi_{12}(\boldsymbol{d}y \otimes \boldsymbol{d}x)$$

$$= \Psi_{12}(\boldsymbol{d}x \otimes \boldsymbol{d}y - \boldsymbol{d}y \otimes \boldsymbol{d}x) = \Psi_{12}(\boldsymbol{d}x \wedge \boldsymbol{d}y) \qquad \text{[by (34.3)]}$$

Then by the same reasoning and a simple rearrangement

$$\mathbf{\Psi} = \Psi_{23}(\boldsymbol{d}x \wedge \boldsymbol{d}y) + \Psi_{31}(\boldsymbol{d}x \wedge \boldsymbol{d}z) + \Psi_{12}(\boldsymbol{d}y \wedge \boldsymbol{d}z)$$

which confirms (34.9) and (34.7).

§34.6, p. 378. *It follows [**exercise**] that the flux is actually a 2-form.*

[34.5] depicts $\underline{\mathbf{\Psi}}$ as a vector perpendicular to the plane P defined by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Therefore, the direction of $\underline{\mathbf{\Psi}}$ depends on $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The magnitude $|\,\underline{\mathbf{\Psi}}\,|$ is a constant, the unit 1. Hence, the amount of fluid crossing P per unit time is

$$\mathcal{V} = |\,\underline{\mathbf{\Psi}}\,|\,\mathcal{A}_P = \Psi_{12}\,\mathcal{A}_P = \mathcal{A}_P = (\boldsymbol{d}x \wedge \boldsymbol{d}y)\,(\boldsymbol{v}_1, \boldsymbol{v}_2) = \phi(\boldsymbol{v}_1, \boldsymbol{v}_2),$$

which is a 2-form.

[34.5] suggests that the flow may not be perpendicular to the plane defined by $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, but I think it must be. In paragraphs 2 to 4 of p. 378, we learn why $\boldsymbol{\Phi} = \boldsymbol{\Psi}$, which is a 2-form.

34.8 The Faraday and Maxwell 2-forms.

It is given that

$$\boldsymbol{E} = E_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + E_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + E_z(\boldsymbol{d}x \wedge \boldsymbol{d}y) \qquad (34.18)$$
$$\boldsymbol{\epsilon} = E_x\,\boldsymbol{d}x + E_y\,\boldsymbol{d}y + E_z\,\boldsymbol{d}z \qquad (34.19)$$
$$\boldsymbol{B} = B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + B_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + B_z(\boldsymbol{d}x \wedge \boldsymbol{d}y) \qquad (34.20)$$
$$\boldsymbol{\beta} = B_x\,\boldsymbol{d}x + B_y\,\boldsymbol{d}y + B_z\,\boldsymbol{d}z \qquad (34.21)$$

The Faraday 2-form

$$\boldsymbol{F} = \boldsymbol{\epsilon} \wedge \boldsymbol{d}t + \boldsymbol{B} \qquad (34.22)$$

$$\boldsymbol{F} = \left(E_x\,\boldsymbol{d}x + E_y\,\boldsymbol{d}y + E_z\,\boldsymbol{d}z\right) \wedge \boldsymbol{d}t + \left(B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + B_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + B_z(\boldsymbol{d}x \wedge \boldsymbol{d}y)\right)$$

Let **EMF** be electromagnetic force exerted on charged particle:

$$\textbf{EMF} = \frac{d\pi}{d\tau} = q\boldsymbol{F}(\,\ldots,\, \boldsymbol{u}) \qquad (34.23)$$

The following is an attempt to expand (34.23) using the givens.

The structure of the RHS suggests that **u** is chosen to be the second vector in the wedge product and the vector picked out by the second 1-form of the electrical wedge products. If that is the case, one can insert any **v** to obtain:

$$\boldsymbol{F}(\boldsymbol{v},\, \boldsymbol{u}) = \left[\left(E_x\,\boldsymbol{d}x + E_y\,\boldsymbol{d}y + E_z\,\boldsymbol{d}z\right) \wedge \boldsymbol{d}t\right](\boldsymbol{v},\; \boldsymbol{u}) + \left[B_x(\boldsymbol{d}y \wedge \boldsymbol{d}z) + B_y(\boldsymbol{d}z \wedge \boldsymbol{d}x) + B_z(\boldsymbol{d}x \wedge \boldsymbol{d}y)\right](\mathbf{v}, \mathbf{u})$$

$$= \left(E_x\, v^1 + E_y\, v^2 + E_z\, v^3\right) \wedge u^0 + \left[B_x\left(v^2 \wedge u^3\right) + B_y\left(v^3 \wedge u^1\right) + B_z\left(v^1 \wedge u^2\right)\right]$$

$$= \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} u^0 + \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \cdot \left\{ \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \times \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \right\} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \cdot \begin{pmatrix} u^1\, u^0 \\ u^2\, u^0 \\ u^3\, u^0 \end{pmatrix} + \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \cdot \begin{pmatrix} v^2\, u^3 - v^3\, u^2 \\ v^3\, u^1 - v^1\, u^3 \\ v^1\, u^2 - v^2\, u^1 \end{pmatrix} \quad \text{(see p. 380)}$$

$$\boldsymbol{F}(\boldsymbol{v},\, \boldsymbol{u}) = E_x\, u^1\, u^0 + E_y\, u^2\, u^0 + E_z\, u^3\, u^0 + B_x\left(v^2\, u^3 - v^3\, u^2\right) + B_y\left(v^3\, u^1 - v^1\, u^3\right) + B_z\left(v^1\, u^2 - v^2\, u^1\right)$$

Then **EMF =** $\frac{d\pi}{d\tau} = q\boldsymbol{F}(\boldsymbol{v}, \boldsymbol{u})$.

But why should adding a flux 2-form **B** to a 2-form ($\boldsymbol{\epsilon} \wedge \mathbf{d}t$) and multiplying it by the charge q result in the electromagnetic force? Does ($\boldsymbol{\epsilon} \wedge \mathbf{d}t$) represent the initial state?