Needham, VDGF, Chapter 33, Notes
June 24, 2026

P. 361. **§33.3.1**. [**exercise**] *If **H** and **J** are both of valence* $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$*, then*

$$(\boldsymbol{H} + \boldsymbol{J})\,(\boldsymbol{\varphi},\ \boldsymbol{\psi})\ ||\ \boldsymbol{v})) \equiv \boldsymbol{H}(\boldsymbol{\varphi},\ \boldsymbol{\psi}\ ||\ \boldsymbol{v})) + \boldsymbol{J}(\boldsymbol{\varphi},\ \boldsymbol{\psi}\ ||\ \boldsymbol{v}).$$

*It is easy to check [**exercise**] that (**H** + **J**) is indeed another tensor of valence* $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$.

In 33.2 it is shown how demonstrate that $\boldsymbol{L}(\boldsymbol{\varphi}\ ||\ \boldsymbol{v})$ is a linear function of both $\boldsymbol{\varphi}$ and **v**. We can use the same method to show that

$(\boldsymbol{H} + \boldsymbol{J})\,(\boldsymbol{\varphi},\ \ldots\ ||\ \boldsymbol{v})$ is a real valued function of both $\boldsymbol{\varphi}$ and **v**
$(\boldsymbol{H} + \boldsymbol{J})\,(\ \ldots,\ \boldsymbol{\psi}\ ||\ \boldsymbol{v})$ is a real valued function of both $\boldsymbol{\psi}$ and **v**

by applying the idea of linearity within each of the three slots of tensors **H** and **J**, and finally to the addition of **H** and **J**, one can show that $(\boldsymbol{H} + \boldsymbol{J})$ is a real-valued linear function of $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, and **v** therefore a tensor of valence $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$.

33.4, P. 362. *Thus we see that (2nd box):*

The set of tensors $\left(\boldsymbol{d}x^i \otimes \boldsymbol{d}x^j\right)$ *forms a basis for tensors of valence* $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Note that if we apply this idea to (33.1), then we find that

$$\boldsymbol{\varphi} \otimes \boldsymbol{\psi}(\boldsymbol{v},\ \boldsymbol{w}) = \varphi_i\ \psi_j\left(\boldsymbol{d}x^i \otimes \boldsymbol{d}x^j\right)$$

We show this:

$$\boldsymbol{\varphi} \otimes \boldsymbol{\psi}(\boldsymbol{v},\ \boldsymbol{w}) \equiv \boldsymbol{\varphi}(\boldsymbol{v})\ \boldsymbol{\psi}(\boldsymbol{w}) \qquad (33.1)$$

$$= \boldsymbol{\varphi}\left(v^i\ \boldsymbol{e_i}\right)\ \boldsymbol{\psi}\left(w^j\ \boldsymbol{e_j}\right) \qquad (32.5,\ \ln 5)$$

$$= \boldsymbol{\varphi}(e_i)\ \boldsymbol{\psi}\left(e_j\right) v^i\ w^j \qquad \text{(by linearity, ex. 33.4, ln 8)}$$

$$= \varphi_i\ \psi_j\ v^i\ w^j \qquad \text{(32.5, first box)}$$

$$= \varphi_i\ \psi_j\left[\boldsymbol{d}x^i(\boldsymbol{v})\right]\left[\boldsymbol{d}x^j(\boldsymbol{w})\right] \qquad \text{(32.6.3 and 33.4, ln 10)}$$

$= \varphi_i\, \psi_j\big(\boldsymbol{d}x^i \otimes \boldsymbol{d}x^j\big)\,(\boldsymbol{v},\, \boldsymbol{w}) \qquad (33.4, \ln 11\,)$

Now abstract away from specific vectors, leaving

$\boldsymbol{\varphi} \otimes \boldsymbol{\psi} = \varphi_i\, \psi_j\big(\boldsymbol{d}x^i \otimes \boldsymbol{d}x^j\big)$

P. 364, § 33.6. Needham sees a connection between linear algebra and tensors of valence $\left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}$. He writes a tensor $\boldsymbol{L}(\boldsymbol{\varphi} \,||\, \boldsymbol{v})$, with components $L^i_j = \boldsymbol{L}\big(\boldsymbol{d}x^i \,||\, \boldsymbol{e}_j\big)$. We are used to seeing $T^i_j = \boldsymbol{T}\big(\boldsymbol{e}_i,\, \boldsymbol{e}_j\big)$, or $T_{\text{ij}}\big(\boldsymbol{d}x^i \otimes \boldsymbol{d}x^j\big)$. But $\boldsymbol{L}(\boldsymbol{\varphi} \,||\, \boldsymbol{v})$ has both a 1-form slot and a vector slot. $\boldsymbol{d}x^i$ fills the 1-form slot $\boldsymbol{\varphi}$, and $\boldsymbol{e}_j$ fills the vector slot **v**. It appears that L^i_j returns a set of real numbers. Needham describes how to write these tensor forms given the available structures in linear algebra. The vector terms are written as column vectors and the dual basis 1-forms are written as row vectors:

$\{\boldsymbol{e}_1,\, \boldsymbol{e}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix},\, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$

$\big\{\boldsymbol{d}x^1,\, \boldsymbol{d}x^2\big\} = \{[1,\, 0],\, [0,\, 1]\}$

$\text{dx}^1(\boldsymbol{v}) = [1,\, 0] \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = v^1 \qquad (1)$

Then, seemingly under the assumption that the only possible operation available to **L** is tensor multiplication, Needham writes:

$\boldsymbol{L} = L^i_j\, \boldsymbol{e}_i \otimes \boldsymbol{d}x^j$

Does the fact that that i and j both appear as upper and lower indices mean that the Einstein summation convention applies? Perhaps not, as L^i_j indexes components. Needham finds the outer product $\boldsymbol{e}_i \otimes \boldsymbol{d}x^j$, leaving the components L^i_j unexpanded as unecessary to his purpose of drawing the connection between tensors and linear algebra, or perhaps tacitly as an exercise for the reader.

$$\begin{aligned}
\boldsymbol{L} &= L^1_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1,\, 0] + L^1_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} [0,\, 1] + L^2_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1,\, 0] + L^2_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0,\, 1] \qquad (2) \\
&= L^1_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + L^1_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + L^2_1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + L^2_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{bmatrix}
\end{aligned}$$

Now return to (2), which multiplies a column vector and a row vector *in that order*. This order of multiplication of matrices is not defined in David Poole, *Linear Algebra: A Modern Introduction*. Brooks/Cole, 2003, so perhaps it is not a part of linear algebra. Then it must be a convention of tensor multiplication. The connection to linear algebra seems problematic.

But the more important issue is why the order of multiplication is $\boldsymbol{e}_i \otimes \boldsymbol{d}x^j$, as shown in $\boldsymbol{L} = L^i_j\, \boldsymbol{e}_i \otimes \boldsymbol{d}x^j$. In linear algebra multiplication, the order of the matrices matters.

The reason can be found in a review of notation in **33.4** to **33.6**;

$T_{\text{ij}} = \boldsymbol{T}(e_i, e_j)$, $T = T_{\text{ij}}(\boldsymbol{d}x^i \otimes \boldsymbol{d}x^j)$, $\left\{ \begin{matrix} 0 \\ 2 \end{matrix} \right\}$, p. 362

$\boldsymbol{K}(\varphi, \psi)$, $\boldsymbol{K} = K^{\text{ij}}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$, $\left\{ \begin{matrix} 2 \\ 0 \end{matrix} \right\}$, p. 363

$L^i_j = \boldsymbol{L}(\boldsymbol{d}x^i \parallel \boldsymbol{e}_j)$, $\boldsymbol{L} = L^i_j\, \boldsymbol{e}_i \otimes \boldsymbol{d}x^j$, $\left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}$, *p*. 364

T, K, or L with superscripts or subscripts stand for components.

A basis can be either a 1-form dual basis or a vector basis.

Tensors **T**,**K**, and **L** can all be written as [(components)(basis ⊗ basis)].

If the arguments of a tensor are 1-forms (e.g. φ, $\boldsymbol{d}x^i$), then the bases are vectors (e.g. $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$).

If the arguments of a tensor are vectors (e.g. **v**, $\boldsymbol{e}_i$), then the bases are 1-forms (e.g. $\boldsymbol{d}x^i \otimes \boldsymbol{d}x^j$).

If L is a tensor of valence $\left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}$, i.e. a matrix, then the basis product is (vector ⊗ 1-form) (e.g. $\boldsymbol{e}_i \otimes \boldsymbol{d}x^j$).

The upper and lower positions of superscripts and subscripts on the components (e.g. L^i_j) correspond to their positions on basis arguments (e.g. $\boldsymbol{L}(\boldsymbol{d}x^i \parallel \boldsymbol{e}_j)$ and the upper and lower positions of 1-forms and vectors in valence curly brackets, but these positions are reversed on the basis products and vector components (e.g. v^i). Why is the reversal needed?

Position of index on

tensor component: 1-form **up**, vector **down**, ex. K^{ij}, T_{ij}.

dual basis 1-form: **up**, ex. $\boldsymbol{d}x^i$.

basis vector: **down**, ex. $\boldsymbol{e}_i$.

1-form component: **down**, ex. φ_i, p. 379.

vector component: **up**, ex. v^j.

In the tensor product of $\boldsymbol{L} = L^i_j\, \boldsymbol{e}_i \otimes \boldsymbol{d}x^j$, the 1-form $\boldsymbol{d}x^j$ has become the base for the vector $\boldsymbol{e}_j$ and the vector $\boldsymbol{e}_i$ has become the base for the 1-form $\boldsymbol{d}x^i$. Compare the standard tensor form of the components: $L^i_j = \boldsymbol{L}(\boldsymbol{d}x^i \,||\, \boldsymbol{e}_j)$. The index of the argument transfers to its corresponding base in the product basis.

Pp. 366, § 33.7. Proof that $\boldsymbol{A} \otimes \boldsymbol{B}$ is *another tensor , independent of the specific* $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{d}x^i\}$ *that gave rise to the components being summed*.

First observe that $\{\boldsymbol{e}_j\}$ is a list of basis vectors. $\{\boldsymbol{d}x^i\}$ is a list of components of the dual basis. The first step is to prove an equivalence between components (not between tensors) by contraction:

$$A^{\mathrm{ij}}\, B_{\mathrm{jk}} \equiv C^i{}_k$$

Once the components are known to be equivalent, one shows that there is are 1-form components φ_i and vector components v^k such that

$$\boldsymbol{C}(\boldsymbol{\varphi} \,||\, \boldsymbol{v}) = C^i{}_k\, \varphi_i\, v^k.$$

Needham wrote: To prove this, first observe that

$$C^i{}_k\, \varphi_i\, v^k = A^{\mathrm{ij}}\, B_{\mathrm{jk}}\, \varphi_i\, v^k$$

We can write A^{ij} because **A** has valence $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and B_{jk}, because **B** has valence $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. This suggests there are two 1-forms associated with A and two vectors associated with B. Then, replace the components with the full tensor forms together with their respective bases: $A^{\mathrm{ij}} \longrightarrow \boldsymbol{A}(\boldsymbol{d}x^i, \boldsymbol{d}x^j)$, $B_{\mathrm{jk}} \longrightarrow \boldsymbol{B}(\boldsymbol{e}_j, \boldsymbol{e}_k)$. (This seems to contradict Needham's rule in the NOTATIONAL WARNING, p. 367, that 1-forms are down and vectors are up. **A** takes 1-form arguments $\boldsymbol{d}x^i$ and $\boldsymbol{d}x^j$, but they are indexed up like a vector, and **B** takes vector arguments, but they are indexed down like a 1-form. Do the rules apply only from p. 367? It is confusing, but consider $\boldsymbol{e}_j$. It is a vector, but $\boldsymbol{e}_1$ has components $\{\boldsymbol{e}_1{}^1, \boldsymbol{e}_1{}^2, \ldots, \boldsymbol{e}_1{}^n\}$. I can't explain the indexing of the $\boldsymbol{d}x^i$ 1-forms except for the fact that the dual basis has i components.)

The 1-forms φ_i and the vector components v^k are real numbers, so they can be rearranged by the linearity of tensors. This is where experience with tensor contractions must play a role. The indexed set of 1-forms φ_i are taken into the **A** tensor to create an Einstein summation with the first dual basis $\boldsymbol{d}x^i$. The vector components are taken into the **B** tensor to create an Einstein summation with the second set of basis vectors $\boldsymbol{e}_k$. Since the 1-form $\boldsymbol{\varphi}$ has only one vector argument, instead of writing $\boldsymbol{A}(\boldsymbol{\varphi}(\boldsymbol{d}x^i), \boldsymbol{d}x^j)$, one writes $\boldsymbol{A}(\boldsymbol{\varphi}, \boldsymbol{d}x^j)$. Instead of writing $\boldsymbol{B}(\boldsymbol{e}_j, v^k\, \boldsymbol{e}_k)$, one can write $\boldsymbol{B}(\boldsymbol{e}_j, \boldsymbol{v})$.

Then Needham uses the linearity of **A** to move the real number $\boldsymbol{B}(\boldsymbol{e}_j, \boldsymbol{v})$ inside the parentheses of **A** and create an Einstein summation with the dual basis $\boldsymbol{d}x^j$. It appears that he has omitted to write $\boldsymbol{B}(\boldsymbol{e}_j, \boldsymbol{v})\, \boldsymbol{d}x^j = \boldsymbol{B}(\boldsymbol{d}x^j\, \boldsymbol{e}_j, \boldsymbol{v})$ and has instead written $\boldsymbol{B}(\boldsymbol{e}_j, \boldsymbol{v})\, \boldsymbol{d}x^j = \boldsymbol{B}(\, ..., \boldsymbol{v})$. $\boldsymbol{d}x^j$ requires a vector. By leaving $\boldsymbol{d}x^j$ outside the parentheses of **B** and writing the ellipses instead, it is perhaps easier to see that a vector is required to make $\boldsymbol{B}(\boldsymbol{e}_j, \boldsymbol{v})\, \boldsymbol{d}x^j$ into a 1-form.

Then, $\boldsymbol{C}(\boldsymbol{\varphi} \,||\, \boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{\varphi}, \boldsymbol{B}(\, ..., \boldsymbol{v})$, and it is easy to see that

$$C^i{}_k\, \varphi_i\, v^k = \boldsymbol{C}(\boldsymbol{\varphi} \,||\, \boldsymbol{v})$$

P. 366, § 33.8. Changing Valence with the Metric Tensor

The valence of the metric tensor is $\left\{ \begin{matrix} 0 \\ 2 \end{matrix} \right\}$, indicating that it has two slots for vectors.

In paragraph 2, Needham appears to describe a constant vector **n** paired with a variable vector **w**, using the expression “if we leave one slot of **g** open,” presumably for **w**. Is this because tensor analysis distinguishes between 1-forms, functions, and vectors, so we speak of “slots” for 1-forms or vectors, rather than arguments of a tensor? Note that both tensors and 1-forms are functions, though of different kinds.

P. 367. NOTATIONAL WARNING. We are warned to be careful. Note first that by universal convention, *the* ***components*** *of the* ***1-form corresponding to*** *the vector* ***n*** *are denoted* n_i, *violating our Greek/Roman dichotomy between 1-forms and vectors.*

Since the orthographic rule has been violated, he resorts to the up or down position of indices to indicate whether a component belongs to a vector (up) or a 1-form (down). So n_i is a 1-form component and n^i is a vector component.

The 1-form, designated $\boldsymbol{\nu}$ (pronounced nu) has components $n_i = g_{\text{ij}}\, n^j$. The 1-form components are multiplied by the dual basis components to obtain $\boldsymbol{\nu} = n_i\, \boldsymbol{d}x^i$ in an Einstein summation. One can also write $n_i = \boldsymbol{\nu}(\boldsymbol{e}_i) = \boldsymbol{g}(\boldsymbol{e}_j, \boldsymbol{n})$.

Notice that the index on $\boldsymbol{d}x^i$ is up. This marks it as indexing a vector component. We gather that the vector itself is

$$\begin{pmatrix} dx^1 \\ dx^2 \\ \ldots \end{pmatrix}$$

But what of the n_i? By the universal convention, it is a component of the 1-form $\boldsymbol{v}$, rather than the vector **n**. The Einstein summation convention gives:

$$\boldsymbol{v} = n_1\, \boldsymbol{d}x^1 + n_2\, \boldsymbol{d}x^2 + \ldots.$$

Showing that $\boldsymbol{v}$ really is a 1-form (real function).

P. 367. *...if* $\mathbf{n} = n^j\, \mathbf{e}_j$ *is a vector in Minkowski spacetime, with metric (30.6), then [**exercise**] the corresponding 1-form is*

$$\boldsymbol{v} = n_i\, \boldsymbol{d}x^i = n^0\, \boldsymbol{d}t - n^1\, \boldsymbol{d}x - n^2\, \boldsymbol{d}y - n^3\, \boldsymbol{d}z.$$

(30.6): $ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) = [dx^0]^2 - ([dx^1]^2 + [dx^2]^2 + [dx^3]^2)$

The square brackets merely set off the indices from the powers of two.

It seems clear from the discussion of notation (above) that $\boldsymbol{d}x^i$ is being substituted for dx, dy, dz in (30.6), or for dx^i with indices in {0, 1, 2, 3} and dx^i is shown to consist of a time dimension and three negative space dimensions. And what is **n**, exactly? The bold letter **n** suggests a vector normal to the dual basis.

v appears to represent the inner product of the vector **n** and the dual basis $\{\boldsymbol{d}x^i\}$. The dual basis picks out the components of a vector (**w** in (33.3)) on the space-time coordinates. The inner product produces the projection of **w** onto **n**.

By analogy with [32.9], $\boldsymbol{v}$ could be a stack with normal vector $\begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{pmatrix}$

and spacing $\frac{1}{\sqrt{(n_0)^2+(n_1)^2+(n_2)^2+(n_3)^2}}$.

From p. 314: $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u} = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u})$.

From p. 315: $ds^2 = \boldsymbol{g}(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}) = \boldsymbol{g}(dx^i\, \boldsymbol{e}_i, dx^j\, \boldsymbol{e}_j) = \boldsymbol{g}(\boldsymbol{e}_i, \boldsymbol{e}_j)\, dx^i\, dx^j = g_{ij}\, dx^i\, dx^j$

$g_{ij} \equiv \boldsymbol{g}(\boldsymbol{e}_i, \boldsymbol{e}_j)$

$$g_{00} = +1, \;\; g_{11} = g_{22} = g_{33} = -1$$

One can write the 1-form by finding the inner product of the components and the dual basis:

$$\boldsymbol{v} = n_i \, \boldsymbol{d}x^i, \text{ where } n_i = g_{\text{ij}} \, n^j = \begin{pmatrix} +n^0 \\ -n^1 \\ -n^2 \\ -n^3 \end{pmatrix}$$

$$\boldsymbol{v} = n_i \, \boldsymbol{d}x^i = \begin{pmatrix} +n^0 \\ -n^1 \\ -n^2 \\ -n^3 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{d}t \\ \boldsymbol{d}x \\ \boldsymbol{d}y \\ \boldsymbol{d}z \end{pmatrix} = n^0 \, \boldsymbol{d}t - n^1 \, \boldsymbol{d}x - n^2 \, \boldsymbol{d}y - n^3 \, \boldsymbol{d}z$$