Needham, Chapter 32, Notes
G. Palmer, December 11, 2025, 2:10 pm, PDT

P. 347.

$$(\boldsymbol{\omega} + \boldsymbol{\varphi})(\boldsymbol{v}) \equiv \boldsymbol{\omega}(\boldsymbol{v}) + \boldsymbol{\varphi}(\boldsymbol{v})$$

It is easy to check ***[exercise]*** *using (32.2) and (32.3) that this sum is itself a 1-form.*

By (32.2)

$$(\boldsymbol{\omega} + \boldsymbol{\varphi})(\boldsymbol{v}_1 + \boldsymbol{v}_2) \equiv (\boldsymbol{\omega} + \boldsymbol{\varphi})(\boldsymbol{v}_1) + (\boldsymbol{\omega} + \boldsymbol{\varphi})(\boldsymbol{v}_2)$$

$$\equiv \boldsymbol{\omega}(\boldsymbol{v}_1) + \boldsymbol{\omega}(\boldsymbol{v}_2) + \boldsymbol{\varphi}(\boldsymbol{v}_1) + \boldsymbol{\varphi}(\boldsymbol{v}_2)$$

By (32.3)

$$(\boldsymbol{\omega} + \boldsymbol{\varphi})(k\boldsymbol{v}) \equiv \boldsymbol{\omega}(k\boldsymbol{v}) + \boldsymbol{\varphi}(k\boldsymbol{v})$$

$$\equiv k\boldsymbol{\omega}(\boldsymbol{v}) + k\boldsymbol{\varphi}(\boldsymbol{v}) \equiv k[\boldsymbol{\omega}(\boldsymbol{v}) + \boldsymbol{\varphi}(\boldsymbol{v})] \equiv k(\boldsymbol{\omega} + \boldsymbol{\varphi})(\boldsymbol{v})]$$

Needham states that the space of 1-forms is dual to the space of vector arguments because "there is a symmetrical relationship between these two spaces." He gives an example:

$$\boldsymbol{v}(\boldsymbol{\omega}) \equiv \boldsymbol{\omega}(\boldsymbol{v})$$

From the NOTES in 32.2, p. 346, 1-forms are presented in lowercase bold Greek letters and vectors in lowercase Roman letters. The equation above seems to say that a vector **v**, with n components, is equivalent to $\boldsymbol{\omega}$, which is a 1-form and hence defined as a linear, real-valued function. It appears to be a contradiction, but perhaps one just has to accept that ***v*(_)** is defined as a real-valued 1-form. Perhaps one can think of it as a construction combining the vector **v** with the dot product operator. But then, by symmetry, one would also have to define $\boldsymbol{\omega}$ as a vector, as in § **32.3.4**, last line.

Let $\boldsymbol{v} = v_1\,\boldsymbol{e}_1 + v_2\,\boldsymbol{e}_2$ and let $\boldsymbol{\omega} = \{\omega_1,\ \omega_2\}$. Then

$$\boldsymbol{\omega}(\boldsymbol{v}) = \omega^1(\boldsymbol{v})\,\boldsymbol{e}_1 + \omega^2(\boldsymbol{v})\,\boldsymbol{e}_2 = \left\{v^1\,\boldsymbol{e}_1 + v^2\,\boldsymbol{e}_2\right\} = \boldsymbol{v}$$

$$\omega_1\,\boldsymbol{e}_1(v) + \omega_2\,\boldsymbol{e}_2(v)$$

P. 347. Dual Bases are introduced in Section **32.2**. In an attempt to understand them I went to the Wikipedia page, where I found an example, which I have adapted and expanded a bit: *If V is R^2, let its* *basis be chosen as* $\{\mathbf{e}_1 = (1/2,\ 1/2),\ \mathbf{e}_2 = (0,\ 1)\}$. Then $\mathbf{e}_1$ and $\mathbf{e}_2$, and the super scripted dual bases $\mathbf{e}^1$ and $\mathbf{e}^2$, are one-forms that satisfy Kronecker's Delta, i.e. $\mathbf{e}_1(\mathbf{e}^1) = 1$, $\mathbf{e}_1(\mathbf{e}^2) = 0$, ...

The problem of finding the dual bases can be expressed as a system of equations in matrix notation:

$$\begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} \mathbf{e}^{11} & \mathbf{e}^{21} \\ \mathbf{e}^{12} & \mathbf{e}^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Notice that the original (primal) bases place the vectors in rows. The dual basis vectors, given with superscripts are presented in columns (i.e. transposed). Then, matrix multiplication gives:

$$\begin{pmatrix} \frac{\mathbf{e}^{11}}{2} + \frac{\mathbf{e}^{12}}{2} & \frac{\mathbf{e}^{21}}{2} + \frac{\mathbf{e}^{22}}{2} \\ \mathbf{e}^{12} & \mathbf{e}^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives four equations with four unknowns:

$$\frac{\mathbf{e}^{11}}{2} + \frac{\mathbf{e}^{12}}{2} = 1$$

$$\frac{\mathbf{e}^{21}}{2} + \frac{\mathbf{e}^{22}}{2} = 0$$

$$\mathbf{e}^{12} = 0$$

$$\mathbf{e}^{22} = 1$$

Solve for the dual components, placing the resultant vectors in rows

$$\begin{pmatrix} \mathbf{e}^{11} & \mathbf{e}^{12} \\ \mathbf{e}^{21} & \mathbf{e}^{22} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

or

$$\mathbf{e}^1 = (2,\ 0),\ \ \mathbf{e}^2 = (-1,\ 1)$$

Then

$$(1/2,\ 1/2).\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1$$

$$(1/2,\ 1/2).\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$$

$$(0,\ 1).\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 0$$

$$(0,\ 1).\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

thus satisfying Kronecker's Delta, i.e. $\boldsymbol{e}_i(\boldsymbol{e}^j) = \left\{ \begin{matrix} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{matrix} \right\}$

P. 355. §**32.6.2**. In this section *The Gradient as a 1-Form:* ***d****f*. In the box, Needham wrote:

$$\boldsymbol{d}f(\boldsymbol{v}) \equiv \nabla_{\boldsymbol{v}} f$$

The RHS has the look of a vector. It is represented in bold font. In fact, on p 354, ∇f is shown defined as a vector.

$$\nabla f \equiv \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}$$

Then on p. 355,

$$\delta \text{f} = (\partial_x f)\,\delta \text{x} + \left(\partial_y f\right)\delta \text{y} = (\nabla f) \bullet \boldsymbol{v}$$

So it appears that ∇f is a 1-form, because it takes the vector **v** as an argument and returns a scalar. Further proof is given on p. 356. Then, since one can apply any vector in a dot product, perhaps all vectors are 1-forms when the situation calls for it.

Out[24]=

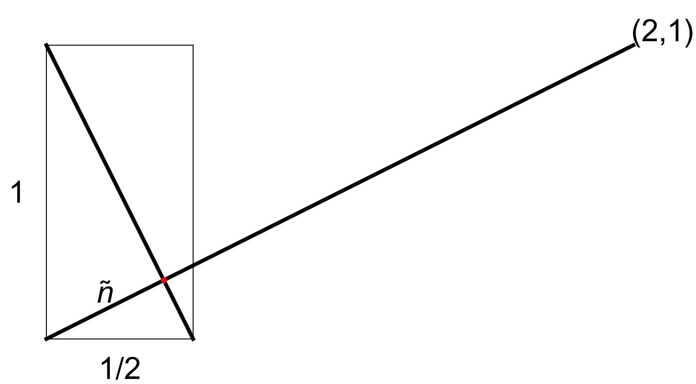

Figure 1. First stack of [32.2]

P. 358. *First note* ***[exercise]*** *that the unit vector normal is*

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}.$$

By inspection of [32.9], there is a normal **n** to a line of the stack $\tilde{\boldsymbol{\varphi}}$ built on the vertical and horizontal stacks by connecting the corners of the boxes created by the horizonal and vertical stacks. $\boldsymbol{n} = \begin{bmatrix}4\\2\end{bmatrix}$ passes through the origin and the point $\begin{bmatrix}4\\2\end{bmatrix}$. Then the unit normal is $\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} = \frac{1}{\sqrt{20}}\begin{bmatrix}4\\2\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}$. This corresponds to the coefficients of the stack formula if $\boldsymbol{\varphi} = 2\,\mathrm{dx} + \mathrm{dy}$ describes a normal $\boldsymbol{n} = \begin{pmatrix}2\\1\end{pmatrix}$.

*...the direction of the stack, which [**exercise**] must have positive components along both of the stacks that are being added.*

Frankly, unless Needham is referring specifically to the stack in [32.9], I do not see why the stack must have positive components along both of the stacks being added. One could imagine stacks with one positive and one negative component, or two negative components.

*But **[exercise]** the spacing of $\tilde{\boldsymbol{\varphi}}$ is $1\big/\sqrt{5}$ and so the number of lines of the stack $\tilde{\boldsymbol{\varphi}}$ pierced by **v** is*

$$\tilde{\boldsymbol{\varphi}}(\boldsymbol{v}) = \textit{number of lines pierced} = \frac{\boldsymbol{v}\cdot\hat{\boldsymbol{n}}}{1/\sqrt{5}} = 2\,a + b = [2\,\mathbf{d}x + \mathbf{d}y]\begin{bmatrix}a\\b\end{bmatrix} = \boldsymbol{\varphi}(\boldsymbol{v}).$$

The spacing $\tilde{\mathbf{n}}$ *of* $\tilde{\boldsymbol{\varphi}}$ can be calculated as the projection of value on the x-axis of the side of small triangle at the origin onto $\hat{\boldsymbol{n}}$.

$$\tilde{\mathbf{n}} = \frac{\left([2,1]^{\mathsf{T}} \cdot [1/2,0]^{\mathsf{T}}\right)[2,1]^{\mathsf{T}}}{\sqrt{2^2+1^2}}$$

$$= \frac{[2,1]^{\mathsf{T}}}{\sqrt{5}}$$

This shows that the direction of $\tilde{\boldsymbol{n}}$ *is* $[2, 1]^T$ *and the length* $|\tilde{\boldsymbol{n}}| = \frac{1}{\sqrt{5}}$. *This is the spacing that Needham specified for* $\boldsymbol{\varphi}$.

One can see from [32.9] that any vector from the origin that touches a stack line that intersects **n**, will pierce (i.e. intersect) the same number of stack lines $\tilde{\boldsymbol{\varphi}}(\boldsymbol{v})$ as pierced by the normal up to and including the point where it touches the stack line. This number is

$$\tilde{\boldsymbol{\varphi}}(\boldsymbol{v}) = \frac{\boldsymbol{v}\cdot[2,1]^T/\sqrt{5}}{1/\sqrt{5}}.$$

For example, say that $\boldsymbol{v} = [7, 3]^T$. Then the number of lines pierced by **v** is

$$\tilde{\boldsymbol{\varphi}}(\boldsymbol{v}) = \frac{[7/2,3]^T \cdot [2,1]^T / \sqrt{5}}{1/\sqrt{5}} = 2 \cdot 7/2 + 1 \cdot 3 = 10 = (2\,\boldsymbol{d}x + \boldsymbol{d}y)[7/2, 3]^T = \boldsymbol{\varphi}\left([7/2, 3]^T\right) = \boldsymbol{\varphi}(\boldsymbol{v})$$

The number 10 is correct. as one can readily check against [32.9]. We conclude that $\tilde{\boldsymbol{\varphi}}(\boldsymbol{v}) = \boldsymbol{\varphi}(\boldsymbol{v})$.

It is easy ***[exercise]*** *to generalize these arguments to confirm that this geometrical construction works for pdx + qdy, yielding a stack with*

normal vector $\begin{bmatrix} p \\ q \end{bmatrix}$ *and spacing* $\frac{1}{\sqrt{p^2+q^2}}$.

Out[ ]=

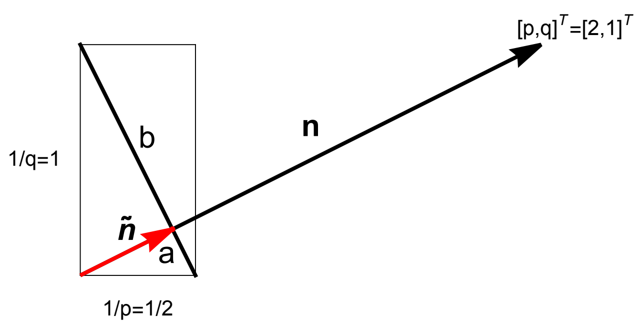

Figure 2. φ=pdx+qdy

A generalization of the spacing in 2D:
Figure 1 zooms in on the lower square of [32.9]. The two points that are joined by the left-tilting stack line are $[1/2,\ 0]^T$ and $[0,\ 1]^T$. One can find the length of the separation between stacks by finding the distance on the normal from the origin to the first stack line, the one that joins $[1/2,\ 0]^T$ and $[0,\ 1]^T$. From Figure 2, one can see that $\tilde{\boldsymbol{n}}$ can be calculated as follows:

$$\tilde{n} = \frac{\left([p,q]^T \cdot [1/p,0]^T\right) [p,q]^T}{\sqrt{p^2+q^2}}$$

$$= \frac{[p,q]^T}{\sqrt{p^2+q^2}}$$

This shows that the direction of $\tilde{\mathbf{n}}$ *is* $[p, q]^T$ *and the length* $|\,\tilde{\mathbf{n}}\,| = \frac{1}{\sqrt{p^2+q^2}}$.

Indeed, in $\mathbb{R}^3$*, we may* ***[exercise]*** *construct* $p\,\mathbf{d}x + q\,\mathbf{d}y + r\,\mathbf{d}z$ *by superimposing the three orthogonal stacks of planes, constructing a new stack of planes passing through the intersections with*

normal vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ *and spacing* $\frac{1}{\sqrt{p^2+q^2+r^2}}$.

Three orthogonal stacks of planes are parallel to xy, yz, and xz spaced at distances of 1/p, 1/q, and 1/r, respectively, along these axes. When the three orthogonal stacks are superimposed, they compose a stack of boxes. For any box, a plane can be passed through three points of intersection which define the plane. Then additional parallel planes can be constructed by multiplying $[p, q, r]^T$ by multiples of the spacing. The normal vector and the unit normal are

$\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ and $\frac{1}{\sqrt{p^2+q^2+r^2}} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$.

The spacing is $\frac{1}{\text{norm}} = \frac{1}{\sqrt{p^2+q^2+r^2}}$.

For illustration, see the answer to Exercise 2, p. 465.

A general proof in three dimensions should work the same way as in two. We project 1/p, 1/q, or 1/r onto the unit normal, which is $[p, q, r]^T$.

$$\tilde{\mathbf{n}} = \frac{\left([p,q,r]^T \bullet [1/p,0,0]^T\right)[p,q,r]^T}{\sqrt{p^2+q^2+r^2}}$$

$$= \frac{[p,q,r]^T}{\sqrt{p^2+q^2+r^2}}$$

This shows that the direction of $\tilde{\mathbf{n}}$ is $[p, q, r]^T$ and the spacing is $|\,\tilde{\mathbf{n}}\,| = \frac{1}{\sqrt{p^2+q^2+r^2}}$.

The number of stack lines pierced by any vector **v** is:

$\tilde{\boldsymbol{\varphi}}(\mathbf{v})$ *number of lines pierced* $= \frac{\mathbf{v} \bullet \tilde{\mathbf{n}}}{1/\sqrt{p^2+q^2+r^2}} = pa + qb + rc = [p\,\mathbf{d}x + q\,\mathbf{d}y + r\,\mathbf{d}z]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boldsymbol{\varphi}(\mathbf{v})$.