Needham, VDGF, Chapter 31, Exercise 9, p. 337.
Palmer, December 1, 2024, 8:45 am, PDT

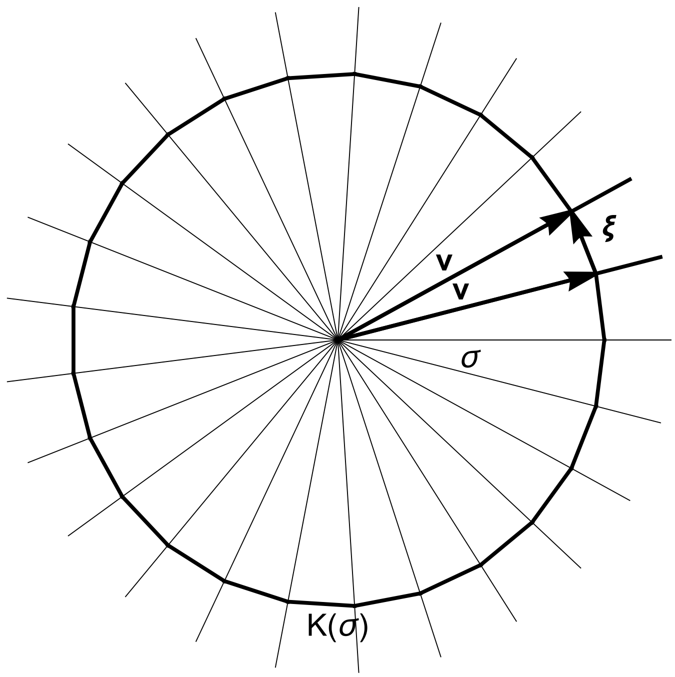
Out[ ]=


Figure 1. Geodesic circle K
on intrinsic radius σ (after [28.6])

9. Gauss's Lemma via Computation. *As in [28.6], consider two neighboring unit speed geodesics launched from 0 with angular separation $d\theta$, and let* $\boldsymbol{v}$ *denote their unit velocity vectors. Let* $\boldsymbol{\xi}$ *be the connecting vector between the geodesics, connecting two points at the same distance $\sigma = 1$ from o, so that both lie on the geodesic circle $K(\sigma)$. Then, as $\delta\theta \to 0$,* $\boldsymbol{\xi}$ *is tangent to $K(\sigma)$. To establish Gauss's Lemma, (28.4)* [p. 274]*, we must therefore show that* $\boldsymbol{v} \bullet \boldsymbol{\xi} = 0$. *(...$\nabla_{\boldsymbol{v}}$ is the ordinary $\mathbb{R}^3$ derivative...).*

First we note that the segments of K are not geodesics on the surface S and K itself is not a geodesic. K is called a "geodesic circle" because the radii are geodesic lines on S. In the following, all instances of ∇ are to be understood as emboldened.

(i) Explain why $\lim_{\sigma\to 0} \boldsymbol{v} \bullet \boldsymbol{\xi} = 0$.

The statement seems ambiguous between (1) $(\lim_{\sigma\to 0} \boldsymbol{v}) \bullet \boldsymbol{\xi} = 0$, and (2) $\lim_{\sigma\to 0} (\boldsymbol{v} \bullet \boldsymbol{\xi}) = 0$. Assume that (2) is the intended interpretation.

Since $\lim_{\sigma\to 0} |\, \boldsymbol{\xi} \,| = 0$, $\lim_{\sigma\to 0} \boldsymbol{v} \bullet \boldsymbol{\xi} = 0$.

(ii) Deduce that to prove $\boldsymbol{v} \bullet \boldsymbol{\xi} = 0$*, it suffices to show that* $\nabla_{\boldsymbol{v}}[\boldsymbol{v} \bullet \boldsymbol{\xi}] = 0$.

Vasco (P.c. VDGF Forum) has provided the answer to this part. $\nabla_{\boldsymbol{v}}[\mathbf{v}\bullet\boldsymbol{\xi}]$ is a vector derivative on a geodesic with the same direction as the geodesic curve σ by construction: $\boldsymbol{v}$ is defined is a unit vector in the direction of σ.

Then if $\nabla_{\boldsymbol{v}}[\mathbf{v}\bullet\boldsymbol{\xi}] = 0$, $\mathbf{v}\bullet\boldsymbol{\xi}$ must be constant on σ. From (i), we know that $lim_{\sigma\to 0}\,\boldsymbol{v}\bullet\boldsymbol{\xi} = 0$. Then $\mathbf{v}\bullet\boldsymbol{\xi}$ must be constant 0 everywhere on σ. It follows that if $\nabla_{\boldsymbol{v}}[\boldsymbol{v}\bullet\boldsymbol{\xi}] = 0$ then $\boldsymbol{v}\bullet\boldsymbol{\xi} = 0$.

(iii) Explain why $\boldsymbol{v}\bullet\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v} = 0$.

As $\theta \longrightarrow 0$, $\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v}$ points in the tangent direction, orthogonal to $\mathbf{v}$. Hence, $\mathbf{v}\bullet\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v} = 0$.

(iv) Explain why $[\boldsymbol{v}, \boldsymbol{\xi}] = \boldsymbol{0}$.

$$[\mathbf{v}, \boldsymbol{\xi}] = \{\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v} - \nabla_{\boldsymbol{v}}\,\boldsymbol{\xi}\}$$

$\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v}$ and $\nabla_{\boldsymbol{v}}\,\boldsymbol{\xi}$ are parallel.

$$|\nabla_{\boldsymbol{v}}\,\boldsymbol{\xi}| \asymp \sigma\, d\theta \qquad (\text{on } K(\sigma))$$

$$|\nabla_{\boldsymbol{\xi}}\,\mathbf{v}| \asymp \delta\xi \asymp \sigma d\theta \qquad (\text{on } K(\sigma))$$

Since $\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v}$ and $\nabla_{\boldsymbol{v}}\,\boldsymbol{\xi}$ are parallel and of equal magnitude,

$$[\mathbf{v}, \boldsymbol{\xi}] = (\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v} - \nabla_{\boldsymbol{v}}\,\boldsymbol{\xi}) = 0.$$

(v) Using the fact that velocity is a geodesic, deduce that $\boldsymbol{\xi}\bullet\nabla_{\mathbf{v}}\,\mathbf{v} = \mathbf{0}$.

Because velocity is a geodesic, it is parallel transported along itself (p. 286), so $\nabla_{\boldsymbol{v}}\,\boldsymbol{v}$ is constant 0, and it follows that $\boldsymbol{\xi}\bullet\nabla_{\mathbf{v}}\,\mathbf{v} = \mathbf{0}$.

(vi) Combine the three previous results to prove that $\nabla_{\boldsymbol{v}}[\mathbf{v}\bullet\boldsymbol{\xi}] = 0$*...*

The three previous results:

$\mathbf{v}\bullet\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v} = 0$	(iii)
$[\mathbf{v}, \boldsymbol{\xi}] = (\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v} - \nabla_{\boldsymbol{v}}\,\boldsymbol{\xi}) = 0$	(iv)
$\boldsymbol{\xi}\bullet\nabla_{\mathbf{v}}\,\mathbf{v} = \mathbf{0}$	(v)

Since $\mathbf{v}$ is velocity on a geodesic ray of length σ, rewrite $\nabla_{\boldsymbol{v}}[\mathbf{v}\bullet\boldsymbol{\xi}]$:

$$\nabla_{\boldsymbol{v}}[\mathbf{v}\bullet\boldsymbol{\xi}] = \partial_\sigma[\mathbf{v}\bullet\boldsymbol{\xi}]\,\boldsymbol{v} = (\partial_\sigma[\boldsymbol{v}]\bullet\boldsymbol{\xi} + \mathbf{v}\bullet\partial_\sigma\boldsymbol{\xi})\,\boldsymbol{v}$$

$\partial_\sigma[\boldsymbol{v}]\bullet\boldsymbol{\xi} = 0$ from (v), from $\partial_\sigma[\boldsymbol{v}]\,\boldsymbol{v} = \nabla_{\mathbf{v}}\,\mathbf{v}$, and the commutative property of the dot product, so

$$\partial_\sigma[\mathbf{v}\bullet\xi] = \mathbf{v}\bullet\partial_\sigma\xi$$

$$\nabla_{\boldsymbol{v}}[\mathbf{v}\bullet\boldsymbol{\xi}] = (\mathbf{v}\bullet\nabla_{\boldsymbol{v}}\,\boldsymbol{\xi})\,\boldsymbol{v}$$

Since $\nabla_{\boldsymbol{v}}\,\boldsymbol{\xi} = \nabla_{\boldsymbol{\xi}}\,\boldsymbol{v}$ from (iv), and $\mathbf{v}\bullet\nabla_{\boldsymbol{\xi}}\,\boldsymbol{v} = 0$ from (iii), we can write $\mathbf{v}\bullet\nabla_{\boldsymbol{v}}\,\boldsymbol{\xi} = 0$, and it follows that $\nabla_{\boldsymbol{v}}[\mathbf{v}\bullet\boldsymbol{\xi}] = 0$. This establishes Gauss's Lemma, that *the geodesic circle $K(\sigma)$ cuts its radii at right angles*.