Needham, VDGF, Chapter 31, Exercise 8, p. 336.
Palmer, November 15, 2024, 9 am, PDT

8. Jacobi Equation on a General Surface of Revolution*. As in Exercise 22, page 89, imagine a particle traveling along a curve in the (x, y)-plane at unit speed, and let its position at time t be [x(t), y(t)]. Now imagine rotating this plane through angle θ about the x-axis. As θ varies from 0 to 2π, the curve sweeps out a surface of revolution.*

(i) Explain why $\dot{x}^2 + \dot{y}^2 = 1$, where the dot represents the time derivative.

Let **r** be a position vector on the generating curve.

$$\boldsymbol{r}(t) = (x(t),\ y(t))$$

$$\dot{\boldsymbol{r}}(t) = \left(\dot{x}(t),\ \dot{y}(t)\right)$$

Since the particle travels at unit speed,

$$v^2 = 1 = \left| \dot{\boldsymbol{r}}(t) \right|^2 = \dot{x}^2 + \dot{y}^2$$

Out[∘]=

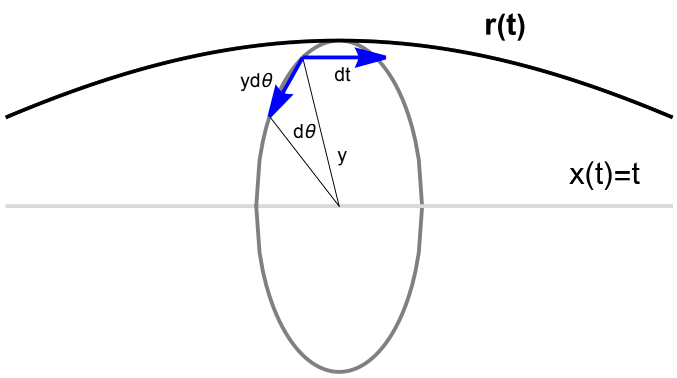

Figure 1. Show geometrically $ds^2 = dt^2 + y^2 d\theta^2$

(ii) Show geometrically that the metric on the surface is $ds^2 = dt^2 + y^2\, d\theta^2$.

See Figure 1. The circle (seen from an angle) lies on a surface of revolution.

(iii) By considering the relative acceleration of two neighboring meridian geodesics, deduce from the Jacobi Equation (28.2) that $\mathcal{K} = -\ddot{y}/y$.

The tiny distance y$d\theta$ is analogous to the separation in [28.3]. One can treat $d\theta$ as a constant and write

$$\left| \boldsymbol{\xi} \right| = \xi \asymp y(t)\, d\theta$$

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} \asymp \dot{y}\,\mathrm{d}\theta$$

$$\frac{d^2\,\xi}{\mathrm{dt}^2} \asymp \ddot{y}\,\mathrm{d}\theta$$

$$\mathrm{d}\theta \asymp \frac{\xi}{y} = \frac{\ddot{\xi}}{\ddot{y}} \Longrightarrow \frac{\ddot{y}}{y} = \frac{\ddot{\xi}}{\xi}$$

Then since it is given in (28.2) that $\ddot{\boldsymbol{\xi}} = -\mathcal{K}\,\boldsymbol{\xi}$, which implies $\mathcal{K} = -\ddot{\boldsymbol{\xi}} \big/ \boldsymbol{\xi}$, and since the $\boldsymbol{\xi}$ vectors and their derivatives remain parallel, we can also write $\mathcal{K} = -\ddot{\xi} \big/ \xi$ from which it follows that $\mathcal{K} = -\ddot{y} \big/ y$.