Needham, VDGF, Chapter 31, Exercise 7, pp. 334-335.
Palmer, November 14, 2024, 3:30 pm, PDT

7. Minding's Theorem. Using the same geodesic polar coordinates as in Section 28.3, the metric takes the form

$$ds^2 = dr^2 + g^2(r)\, d\theta^2$$

so that $g'' = -\mathcal{K} g$.

Using the fact that $g(r) \asymp r$ *(as r vanishes), solve this differential equation in each of the three cases of constant* $\mathcal{K}$*:*

Section 28.3 refers only to "the same geodesic polar coordinates as before."

Prior to launching into specific solutions, it seems worthwhile to point out that the object of this exercise is not crystal clear. If one is given Minding's Theorem, as on p. 21, it is easy to make every deduction in parts (i) to (iii). Then why would one need to solve $g'' = -\mathcal{K}g$ for g? So it must be that the object of the exercise is not to apply the theorem, but to prove it, as pointed out to me by Vasco in personal communication (VDGF Forum), or to find specific instances of it consistent with $\mathcal{K} = 0$, $\mathcal{K} = \left(1/R^2\right)$, and $\mathcal{K} = -\left(1/R^2\right)$. If one can solve for g(r) in each case and substitute the solution into the given general metric to obtain a metric equivalent to known or calculated metrics for the Euclidean plane, sphere, and pseudosphere, then one has proven Minding's Theorem.

There appears to be an error in the problem statement. We find that $g(r) \asymp r$ *(as r vanishes)* to be valid only for parts (i) and (ii). For part (iii), both the algebra and the geometry suggest that $g(r) \asymp R$, the radius of the pseudosphere or other surface of constant negative curvature.

(i) If $\mathcal{K} = 0$ *everywhere, deduce that the surface is locally isometric to the Euclidean plane.*

Since locally, $g(r) \asymp r$ as r vanishes, the solution $g \asymp 0$ is given, r being any geodesic radius from an arbitrary point. Since $g' = r' = 1$ and $g'' = 0$, there is no geodesic curvature on any curve, so the surface is locally Euclidean. Since $g(r) \asymp r$, we can write our local metric (which is everywhere the same) as

$$ds^2 = dr^2 + r^2\, d\theta^2$$

which is the metric for the Euclidean plane. Hence, the surface is locally isometric to the Euclidean plane.

(ii) If $\mathcal{K} = (1/R^2)$ *is constant throughout the surface, deduce that the space* [surface?] *is locally isometric to the sphere of radius R.*

Derive a metric for the sphere. Using [28.3], one can calculate ρ as in the first equation on p. 271, only replacing t with r.

$$\rho = R \sin\theta = R \sin\left(\frac{r}{R}\right)$$

Plotting two orthogonal vectors ds_1 and ds_2 from point p in [28.3], $ds_1 = dr$ and $ds_2 = \rho\, d\theta$. The metric is

$$ds^2 = dr^2 + \rho^2\, d\theta^2$$

$$= dr^2 + \left(R \sin\frac{r}{R}\right)^2 d\theta^2$$

A general solution for a second order linear equation* such as $g'' = -\mathcal{K} g$ *is* $g(r) = A\cos\frac{r}{R} + B\sin\frac{r}{R}$. Since we want g(r) to vanish with r, we let A = 0 and write $g(r) = B\sin\frac{r}{R}$. Then, since we interpret g(r) as a substitution for ρ in [28.3], a distance which vanishes with vanishing r, substitute R for B so that $g(r) = R\sin\frac{r}{R}$. This g(r) satisfies $g(r) \asymp r$ with vanishing r. At $\phi = r/R = \pi/2$, $g(r) = R$. Then $R\sin\frac{r}{R}$ can be substituted for g(r) in the general metric. The result is identical to the metric for the sphere, showing that the surface is locally isometric to the sphere.

(iii) If $\mathcal{K} = -(1/R^2)$ *is constant throughout the surface, deduce that the space* [surface?] *is locally isometric to the pseudosphere of radius R.*

From the metric given in (5.2), p. 53 and illustrated in [5.4][a], we can deduce a metric in polar coordinates (r, θ). Substitute dr for $d\sigma$ (the component on the meridian), $\mathrm{R}e^{-\sigma/R}$ for X (the radius at σ = const.), and $d\theta$ for dx (the horizontal component):

$$d\hat{s}^2 = d\sigma^2 + \left(\mathrm{R}e^{-\sigma/R}\right)^2 dx^2 \qquad \text{(5.2) rearranged}$$

After substitutions, the metric with polar coordinates on the pseudosphere is

$$ds^2 = dr^2 + \left(\mathrm{R}e^{-r/R}\right)^2 d\theta^2$$

Following the same procedure as used by Needham on p. 53 in using (4.10) to calculate $\mathcal{K}$, let $u = r,\ v = \theta,\ A = 1,\ B = \mathrm{R}e^{-r/R}$. Then

$$\mathcal{K} = -\frac{1}{\mathrm{R}e^{-r/R}}\left(\partial_\theta\left[\frac{\partial_\theta[1]}{\mathrm{R}e^{-r/R}}\right] + \partial_r\left[\frac{\partial_r[\mathrm{R}e^{-r/R}]}{1}\right]\right)$$

$$= -\frac{1}{\mathrm{R}e^{-r/R}}\left(0 + \partial_r\left[\mathrm{R}e^{-r/R}(-1/R)\right]\right) = -\frac{1}{\mathrm{R}e^{-r/R}}\left(-\partial_r\left[e^{-r/R}\right]\right) =$$

$$= -\frac{1}{\mathrm{R}e^{-r/R}}\left(-e^{-r/R}(-1/R)\right) = -\frac{1}{R^2}$$

This is the $\mathcal{K}$ we wanted, so we can have some confidence in the metric with polar coordinates. Note that the B of (4.10) is a different constant from the B in the general solution to the differential equation $g'' = -\mathcal{K}$g.

The general solution we calculated for ($g'' = -\mathcal{K}$g) is $g(r) = A\cos\frac{r}{R} + B\sin\frac{r}{R}$. There is a special solution* for $\mathcal{K} = -(1/R^2)$ using real exponents: $g(r) = \mathrm{A}e^{\frac{r}{R}} + \mathrm{B}e^{\frac{-r}{R}}$. Since the second term looks structurally similar to what we need for g(r), we guess that A = 0 and $g(r) = \mathrm{B}e^{\frac{-r}{R}}$. Furthermore, $\mathrm{A}e^{\frac{r}{R}} \to A$ as $r \to 0$, which would mean that g(r) would not vanish with vanishing r unless it were equal to $-B$. Since a constant is needed to substitute for B and the only constant in our construction is R, we write $g(r) = \mathrm{R}e^{\frac{-r}{R}}$. Think of g(r) as the radius of the pseudosphere at any point σ = r. Then it is obvious that g(r) vanishes as $r \to \infty$, and $g(r) \to R$ with vanishing r.

Substituting $g(r) = \mathrm{R}e^{\frac{-r}{R}}$ into the given general metric, one sees it matches the metric for the pseudosphere with polar coordinates. Hence, a surface of constant negative curvature $\mathcal{K} = -(1/R^2)$ is locally isometric to the pseudosphere of radius R.

The three exercises together prove the validity of Minding's Theorem and expand it beyond the sphere to surfaces of constant zero curvature and constant negative curvature, such as the Euclidean plane and the pseudosphere.

Endnote. ***A general solution to*** $\boldsymbol{g'' = -\mathcal{K}g}$ (p. 25, Piaggio, H.T.H., *An Elementary Treatise on Differential Equations*, 1949. London: G. Bell and Sons, Ltd.)

This is a 2nd order linear equation with constant coefficients with the form

$$p_0\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2\,y = 0$$

Then, for $g'' = -\mathcal{K}g$, $p_0 = 1$, $p_1 = 0$, and $p_2 = \mathcal{K} = \frac{1}{R^2}$.

Piaggio suggested a solution of $y = \mathrm{A}e^{\mathrm{mx}}$. An equivalent equation is $g = \mathrm{A}e^{\mathrm{mr}}$. Piaggio replaced the original equation with $\mathrm{A}e^{\mathrm{mx}}(p_0\,m^2 + p_1\,m + p_2) = 0$. Correspondingly, we write $\mathrm{A}e^{\mathrm{mr}}(m^2 + \mathcal{K}) = 0$. Since we don't want $A = 0$, we find that $m = \pm i\sqrt{\mathcal{K}} = m \pm \frac{i}{R}$. Then a general solution is $g(r) = \mathrm{A}e^{\frac{\mathrm{ir}}{R}} + \mathrm{B}e^{-\frac{\mathrm{ir}}{R}}$, but we don't want solutions with complex numbers. A general solution avoiding them is

$$g(r) = e^{\mathrm{ar}}\left(A \sin \frac{r}{R} + \mathrm{B} \cos \frac{r}{R}\right)$$

where a is the x value of the complex number m = $\pm \frac{\mathrm{i}}{\mathrm{R}}$. Since ar = 0,

$$g(r) = A \cos \frac{r}{R} + B \sin \frac{r}{R}$$

If $\mathcal{K} = -(1/R^2)$, then $m^2 = (1/R^2)$ and the general solution becomes

$$g(r) = \mathrm{A}e^{\frac{r}{R}} + \mathrm{B}e^{-\frac{r}{R}}.$$