Needham, VDGF, Chapter 31, Exercise 6, p. 336.
Palmer, November 13, 2024, 9 am, PDT

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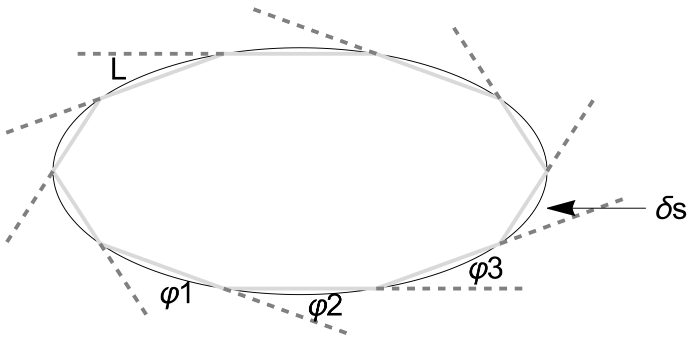

Figure 1. Geodesic m−gon
approaching smooth curve L

(i) By approximating a smooth, closed loop L with a geodesic m-gon, then letting $m \to \infty$, deduce that the holonomy formula (24.6) becomes

$$\mathcal{R}(L) = 2\pi - \oint_L \kappa_g \, ds$$

where κ_g is the geodesic curvature along L, and s is distance along L.

Equation (24.6) pertains to rotation on a geodesic n-gon, which makes it applicable to the geodesic m-gon in this exercise. The accompanying image [24.2] depicts a general geodesic triangle on a general surface. In this exercise, Needham has apparently placed L and the m-gon on a general surface.

$$\mathcal{R}(P_m) = 2\pi - \sum_{i=1}^{m} \varphi_i \qquad \text{(24.6), p. 247}$$

where $\mathcal{R}$ is the holonomy of the loop, P_m is a geodesic m-gon, and φ_i is an angle between the tangent to segment i of the m-gon at a point p and the parallel-transported vector $\boldsymbol{w}_{\|}$ anchored at p. We only have to show that $\oint_L \kappa_g \, \mathrm{ds} \asymp \sum_{i=1}^{m} \varphi_i$.

We know that κ_g at a point p on L is the change in φ on ds, so we can write $\mathrm{d}\varphi_i = (\kappa_g)\,\mathrm{ds}$. As the sides of the m-gon shrink in size and increase in number to infinity, their sum and shape ultimately equals L. Hence, we can write

$$\sum_{i=1}^{m} \varphi_i \asymp \oint_L \mathrm{d}\varphi \asymp \oint_L \kappa_g \, ds$$

This is what we wanted.

In Figure 1, I have used an ellipse to represent a general curve L on a general surface. The segments of L are non-geodesic curves. The geodesic sides of the m-gon approximate L more closely as the number of sides increases. However, φ does increases with each transition from one side to the next of the n-gon, as is shown more clearly in [24.2].

(ii) If P is a closed "polygon" with external angles φ_i, but with edges that are not geodesic (i.e. $\kappa_g \neq 0$), then show that the generalization of (24.6) is

$$\mathcal{R}(P) = 2\pi - \left[\oint \kappa_g \, ds + \sum_i \varphi_i\right].$$

From *(24.6)*, $\mathcal{R}(P_m) = 2\pi - \sum_{i=1}^{m} \varphi_i$, which describes the result of a velocity vector in parallel transport around a geodesic polygon. The vector rotates by the angle $\sum_{i=1}^{m} \varphi_i$. With a geodesic m-gon where m $\longrightarrow \infty$, inscribed in a smooth, closed loop, the rotation about the loop becomes $\oint_L \kappa_g \, ds$. In this part of the exercise, the curved, non-geodesic sides of P are treated under the integral as though they formed a continuous loop, and their rotation is added to the exterior angles under the sum. Hence, the holonomy of the non-geodesic polygon is

$$\mathcal{R}(P_{\text{ng}}) = 2\pi - \left[\mathcal{R}(P_g) + \mathcal{R}(L_{\text{ng}})\right]$$

or

$$\mathcal{R}(P) = 2\pi - \left[\oint_L \kappa_g \, ds + \sum_i \varphi_i\right], \quad \text{non-geodesic P}$$

where R is rotation of a vector under traversal, $\mathcal{R}$ is holonomy, and g and ng stand for geodesic and non-geodesic.

(iii) If R denotes the interior of P, deduce the ***General Local Gauss-Bonnet Theorem****.*

$$\iint_R \mathcal{K} d\mathcal{A} = 2\pi - \left[\oint_L \kappa_g \, \text{ds} + \sum_i \varphi_i\right]$$

The "local" part of the GLG-B Theorem refers to the fact that the theorem refers to a region of a surface rather than to an entire closed, orientable surface such as a sphere or a donut (p. 167).The first term $\iint_R \mathcal{K} d\mathcal{A}$ refers to the total internal curvature on a region R (p. 336). One must distinguish between the region R and the holonomy angle $\mathcal{R}$. Holonomy herein refers to an angle of rotation under parallel-transport in excess or deficiency compared to 2π.

On p. 248, section 24.4, ¶ 1, we find “the holonomy of a loop measures the total curvature within it.” Then this equation simply substitutes the total internal curvature of a local region enclosed by a loop for the holonomy defined in part (ii). The local region is conceived as a polygon with both geodesic and non-geodesic sides.