Needham, VDGF, Chapter 31, Exercise 5, pp. 334-335.
Palmer, November 12, 2024, 2:30 pm, PDT

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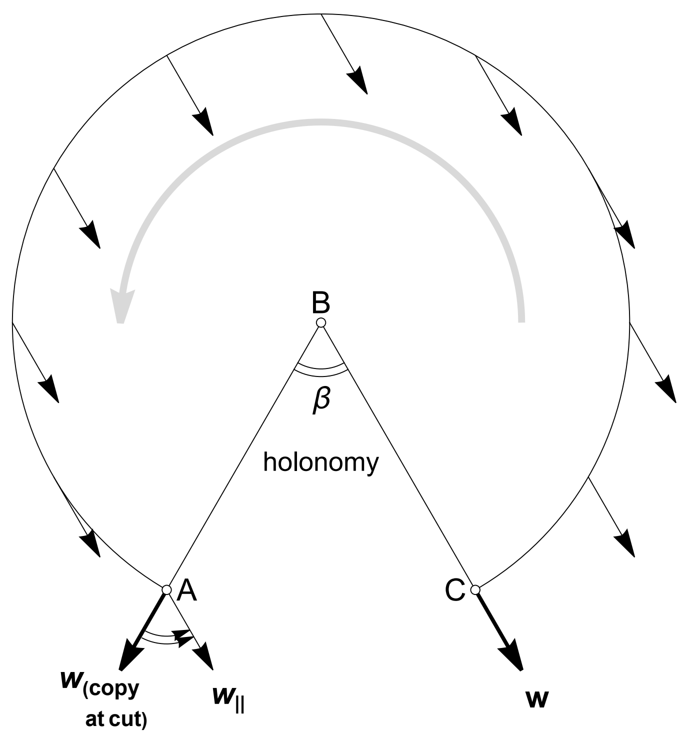

Figure 1a. Parallel transport on flattened cone

5. Holonomy on Cone and Sphere*.*
(i) Reconsider the cone of the semivertical angle α shown in [14.3], page 145. [The curvature $\mathcal{K}$ of the blunted] *tip of the otherwise intrinsically flat cone,* [is] *given by (14.2):*

$$\mathcal{K}\,(\text{spike}) = \beta = \textit{split angle of flattened spike} = 2\,\pi\,(1 - \sin\alpha).$$

By carrying out parallel transport within the flattened cone in [14.3], show that holonomy assigns the same total curvature to the spike.

Most of the illustrations in Chapter 23 show parallel transport along geodesics. In [23.1][b] we see parallel transport of a vector along a general curve. The diagram shows the transported vector maintains constant angles along segments of K, but angles with tangents vary along the curve. All the transported vectors are parallel.

Flattening the cone by cutting along a meridian reveals the angular excess or holonomy induced by removing a pie shaped slice of a flat disc. The length of circumference removed equals β, which equals the angle between AB and BC (because a line between A and the circumference of the disc has length

1). The equation above (14.2) reveals that $\beta = 2\,\pi(1 - \sin\alpha)$. The flat disc can be bent into a cone by bringing the sides of the cut, AB and BC, together, thereby creating the curvature.

In figure 1a, the initial vector **w** at C is parallel-transported anti-clockwise until it reaches A.

A copy of **w** makes an angle of 0 with the cut line AB and $-\pi/2$ with the circular curve, because the flattening is a conformal transformation which preserves angles. Rotate the final $\boldsymbol{w}_{||}$ clockwise about point A to bring it to an angle of 0 with the cut line and $-\pi/2$ with the circular curve segment. At that point it is obvious that the holonomy equals β, the angle of the cut, which we know to be $2\,\pi\,(1 - \sin\alpha)$. β has a positive sign, which is appropriate for the construction, because the direction of $\boldsymbol{w}_{||}$ at A is anticlockwise to the direction of the copy of **w** at A, i.e. at the end of the traversal.

To summarize, Figure 1a shows that the holonomy equals β by appealing to the conformal transformation of flattening the cone. **w** and the copy of **w** must make the same angle with the cut line on their respective sides of the cut. $\boldsymbol{w}_{||}$ remains parallel to initial **w**, so the difference from the copy of **w** is the holonomy. One can see by inspection that it equals β.

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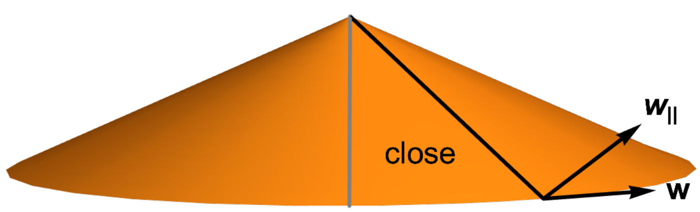

Figure 1b. Displaying holonomy by closing the cut.

In Figure 1b, the cut is closed and the vector $\boldsymbol{w}_{||}$ is now anchored at the same point as **w**. But in closing the cut, $\boldsymbol{w}_{||}$ rotates about the lower end of the cut line by an angle of β. **w** is unchanged. Closing the cut conserves the side length and arc length, which on the cone is the circumference.

One can also show this visually by using what we know to recreate the cone. Bring the edges AB and BC together by rotating AB around B by the angle β (See Figure 1a). When one does this, the vector $\boldsymbol{w}_{||}$ necessarily also rotates about the (moving) point A by the angle β, conserving the angle. When AB coincides with BC, the angle from $\boldsymbol{w}_{||}$ to $\mathbf{w}$ is shown to be β, as in Figure 1b, which shows a cone after closure of a cut. The closure reveals the holonomy in the angle from $\boldsymbol{w}_{||}$ to $\mathbf{w}$ and it shows that the holonomy equals the split angle of flattened spike in Figure 1a, and therefore assigns the same curvature.

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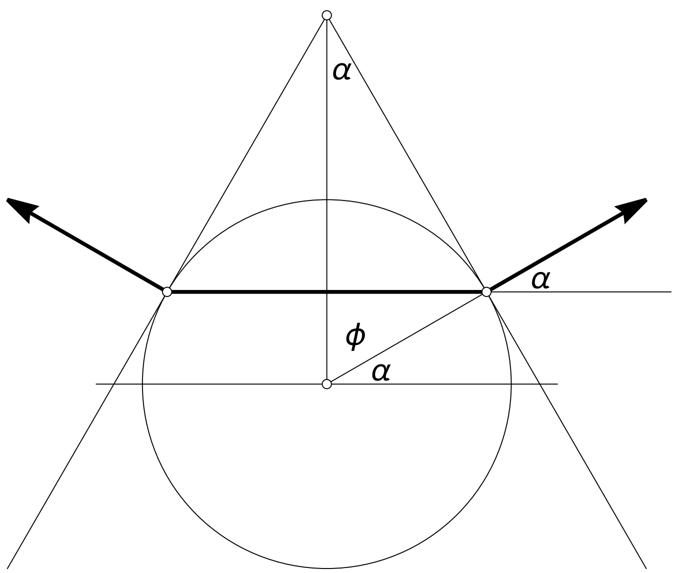


Figure 2. Cone with α and ϕ angles

(ii) On $\mathbb{S}^2$ let us use extrinsic, potato-peeler parallel transport to find the holonomy of a circle of latitude of fixed angle ϕ. Imagine a cone of semi-vertical angle α resting on the sphere so that it touches the sphere along this circle. The strip of peel of the sphere along this circle is therefore (ultimately) the same as the strip of the touching cone. Show that $\alpha = \frac{\pi}{2} - \phi$, and use (i) to confirm that the total curvature within the polar cap is indeed the holonomy of the circle of latitude that bounds it.

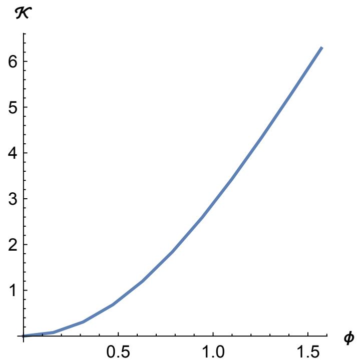
That $\alpha = \frac{\pi}{2} - \phi$ is shown by the labeled angles within the right angle at the centre in Figure 2.

The holonomy of the circle of latitude can be found with the formula (14.3), p. 145, for the curvature of an intrinsic circle: $\mathcal{K} = 2\pi - C(1)$, where $C(r) = 2\pi r \sin\alpha$. Then

$$\mathcal{K} = 2\pi - C(1) = 2\pi - 2\pi\sin\alpha = 2\pi(1 - \sin\alpha)$$

This is equal to the equation for the curvature of the tip of the cone given in part (i).

Figure 3. $\mathcal{K}$=2π(1−cos ϕ)



(iii) On the equator of $\mathbb{S}^2$*, Imagine a vector pointing due east. Now parallel transport it due east along the geodesic equator, so that it returns home seemingly unchanged, i.e. with vanishing holonomy. But this loop encloses half the sphere, with total curvature* 2π*! Use (ii) to reconcile these facts, by gradually increasing* ϕ *from 0 to* $(\pi/2)$*.*

Figure 2 reveals that the area of the polar cap increases with ϕ. Since $\mathcal{K}$ of the peak equals the area of the polar cap, $\mathcal{K}$ should increase with ϕ, as in Figure 3. Write an equation for $\mathcal{K}(\phi)$ and observe the changes (Table 1).

$$\mathcal{K} = 2\,\pi(1 - \sin\alpha) = 2\,\pi\left(1 - \sin\left(\tfrac{\pi}{2} - \phi\right)\right) = 2\,\pi(1 - \cos\phi)$$

The total curvature increases with increasing area from the north pole up to, but seemingly not including, the equator. There the transported vector appears to be unchanged, the holonomy appears to be 0, and the curvature appears to be just the normal curvature $\kappa_n = (1/r)$ in the plane of the equator itself.

From part (ii), the cone is resting on the sphere and $(\alpha = \pi/2 - \phi)$. But by the construction, the cone can only rest on the sphere when $0 < \alpha < \pi/2$. When $\alpha = 0$, the cone is undefined, but one could observe that the cylinder still captures the area of the half-sphere, because it still touches the sphere along the equator. The area of the half-sphere is equivalent to the holonomy. When $\alpha = \pi/2$, the surface degenerates to an Euclidean plane.

To gradually increase ϕ from 0 to $\pi/2$, the length of a meridian of the cone must approach infinity. At $\phi = \pi/2$, $\alpha = 0$ or undefined, and the cone becomes undefined or morphs into a cylinder, depending on one's interpretation.

Vasco (P. c., VDGF) pointed out that the eastward pointing vector on the tangent plane rotates about itself under parallel transport by an angle of $2\,\pi$. On p. 248, section 24.4, ¶ 1, we find "the holonomy of a loop measures the total curvature within it." Since the total curvature of the half-sphere of unit radius is 2π, it makes sense to find holonomy of 2π on the equator. Since a vector that is parallel-transported about the equator undergoes a full rotation of 2π, it is only “seemingly” unchanged.

The holonomy of the equator appears to be extrinsic. We note that rotation on the equator is also rotation about the NS axis, as shown in [24.1]. It would not be visible to local inhabitants on the equator. The geodesic curvature there is zero, so the equator would appear to be straight, as would other tangent lines. Note that total curvature is not the same as geodesic curvature.

Table 1. $\mathcal{K}$ by ϕ for total curvature of tip of cone over polar cap.

ϕ	$\mathcal{K}(\phi)$
0.0000	0.0000
0.1571	0.0774
0.3142	0.3075
0.4712	0.6848
0.6283	1.2000
0.7854	1.8400
0.9425	2.5900
1.1000	3.4310
1.2570	4.3420
1.4140	5.3000
1.5710	6.2830