Needham, VDGAF, Chapter 31, Exercise 4, p. 335.
Palmer, September 13, 2024, 8 am, PT

4. Geodesic Curvature via Intrinsic Differentiation. *Let* $\boldsymbol{v}$ *be the velocity of a particle traveling at unit speed along a trajectory K over the surface* $\mathcal{S}$ *of some peelable fruit. If o is the launch point, let* $(\boldsymbol{e_1}, \boldsymbol{e_2})$ *be an orthonormal basis for* T_o*. Now parallel transport* $\boldsymbol{e_1}$ *along K, and let* $\theta_{||}$ *denote the angle between this parallel-transported* $\boldsymbol{e_1}$ *and* $\boldsymbol{v}$*.*

(i) Explain why $\boldsymbol{e_2}$ *is automatically parallel-transported along K, too:*

$$D_{\boldsymbol{v}}\, \boldsymbol{e_1} = 0 \quad \Longrightarrow \quad D_{\boldsymbol{v}}\, \boldsymbol{e_2} = 0.$$

The angle between $\boldsymbol{e_1}$ and $\boldsymbol{e_1}$ is fixed at $\pi/2$. In ¶ 1, p. 233, it is explained that parallel transport requires that the direction of a vector **w** at each moment is always parallel to its direction a moment before. Since both $\boldsymbol{e_1}$ and $\boldsymbol{e_1}$ lie in the same plane and their angle is fixed, parallel transport of one implies parallel transport of the other.

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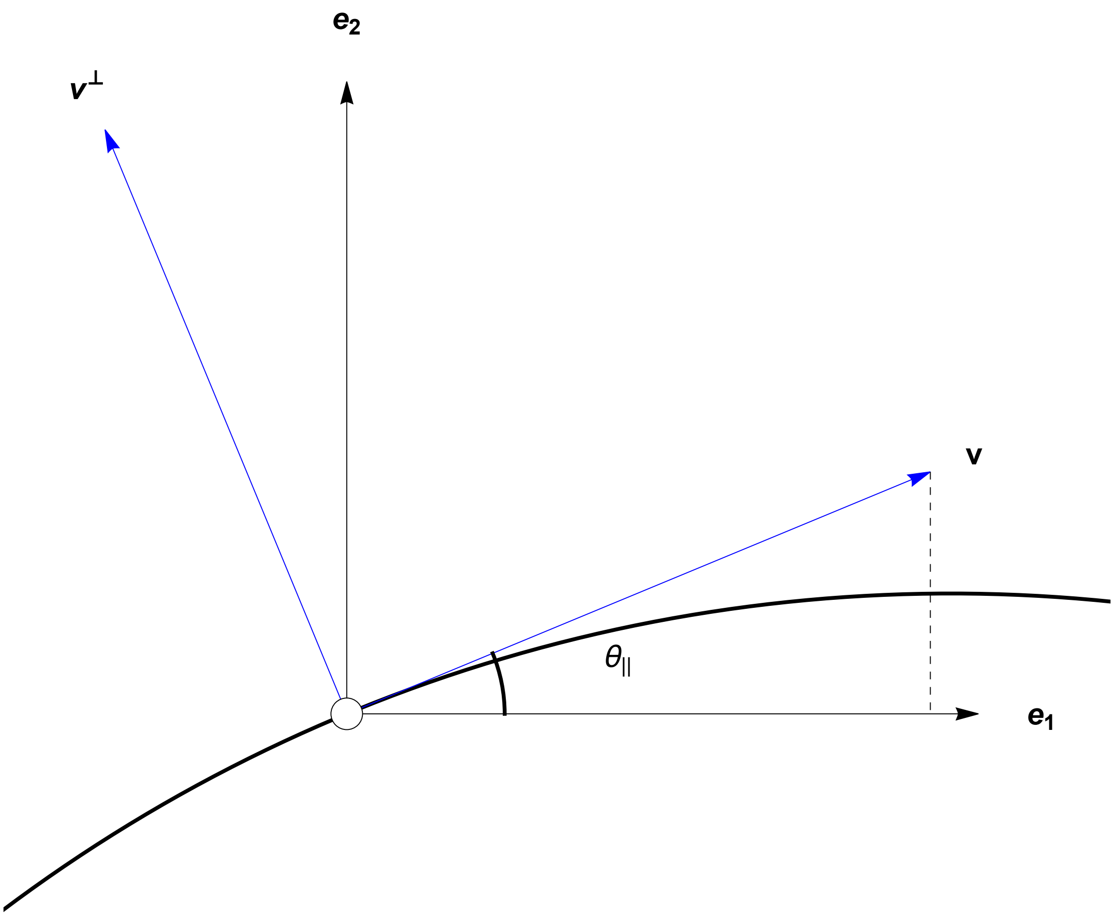

Figure 1. Rotation of vector v by (π/2) within T_p

(ii) Let $\boldsymbol{v}^{\perp}$ be the unit vector obtained by rotating $\boldsymbol{v}$ by (π/2) within T_p. Write $\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$ in terms of ($\boldsymbol{e_1}$, $\boldsymbol{e_2}$). Now use calculation to prove that

$$\boldsymbol{\kappa}_g = D_{\boldsymbol{v}}\, \boldsymbol{v} = \left[D_{\boldsymbol{v}}\, \theta_{||}\right] \boldsymbol{v}^{\perp}.$$

Since we know that the bases ($\boldsymbol{e_1}$, $\boldsymbol{e_2}$) are orthonormal, putting them on the x and y axes should should cause no loss of generality. From part (i), we know that $\boldsymbol{e_1}$ is parallel-transported along K, so $\boldsymbol{e_2}$, which has a fixed angle with $\boldsymbol{e_1}$, must also be parallel-transported. Hence, the frame itself is parallel-transported. From (23.3), p. 244, we know that $\boldsymbol{\kappa}_g = D_{\boldsymbol{v}}\, \boldsymbol{v}$. From Figure 1, let

$\boldsymbol{v} = \cos\theta_{||}\, \boldsymbol{e}_1 + \sin\theta_{||}\, \boldsymbol{e}_2$

$\boldsymbol{v}^{\perp} = -\sin\theta_{||}\, \boldsymbol{e}_1 + \cos\theta_{||}\, \boldsymbol{e}_2$ (because $\cos(\theta_{||} + \pi/2) = -\sin\theta_{||}$ and $\sin(\theta_{||}+\pi/2) = \cos\theta_{||}$).

Then, following the first and third identities on p. 243, and using the results of part (i),

$D_{\boldsymbol{v}}\, \boldsymbol{v} = D_v(\cos\theta_{||}\, \boldsymbol{e}_1 + \sin\theta_{||}\, \boldsymbol{e}_2) = D_v(\cos\theta_{||}\, \boldsymbol{e}_1) + D_v(\sin\theta_{||}\, \boldsymbol{e}_2).$

$= D_v(\cos\theta_{||})\, \boldsymbol{e}_1 + \cos\theta_{||}\, D_v\, \boldsymbol{e}_1 + D_v(\sin\theta_{||})\, \boldsymbol{e}_2 + \sin\theta_{||}\, D_v\, \boldsymbol{e}_2$

$= -\sin\theta_{||}(\theta_{||})'\, \boldsymbol{e}_1 + \cos\theta_{||} \cdot 0 + \cos\theta_{||}(\theta_{||})'\, \boldsymbol{e}_2 + \sin\theta_{||} \cdot 0 = -\sin\theta_{||}(\theta_{||})'\, \boldsymbol{e}_1 + \cos\theta_{||}(\theta_{||})'\, \boldsymbol{e}_2$ (see (i))

$= (\theta_{||})' (-\sin\theta_{||}\, \boldsymbol{e}_1 + \cos\theta_{||}\, \boldsymbol{e}_2) = [D_{\boldsymbol{v}}\, \theta_{||}] (-\sin\theta_{||}\, \boldsymbol{e}_1 + \cos\theta_{||}\, \boldsymbol{e}_2)$

$= [D_{\boldsymbol{v}}\, \theta_{||}]\, \boldsymbol{v}^{\perp}$

I wondered about the significance of the square brackets on the RHS. On p. 243, on the LHS, they serve to indicate “apply” or “∘”. Perhaps curved brackets would have served as well on the RHS: $(D_{\boldsymbol{v}}\, \theta_{||})\, \boldsymbol{v}^{\perp}$.

$\boldsymbol{\kappa}_g = \boldsymbol{D}_{\boldsymbol{v}}\, \boldsymbol{v}$ is proven by the zero value of the surface normal curvature $\boldsymbol{\kappa}_n$ in (23.3), p. 243 and p. 244. Then, since $D_{\boldsymbol{v}}\, \boldsymbol{v} = (D_{\boldsymbol{v}}\, \theta_{||})\, \boldsymbol{v}^{\perp}$, and $\boldsymbol{v}^{\perp}$ is a unit vector, the magnitude $|\, \boldsymbol{\kappa}_g \,|$ must be given by $|\, D_{\boldsymbol{v}}\, \theta_{||} \,|$, thereby proving (23.4).

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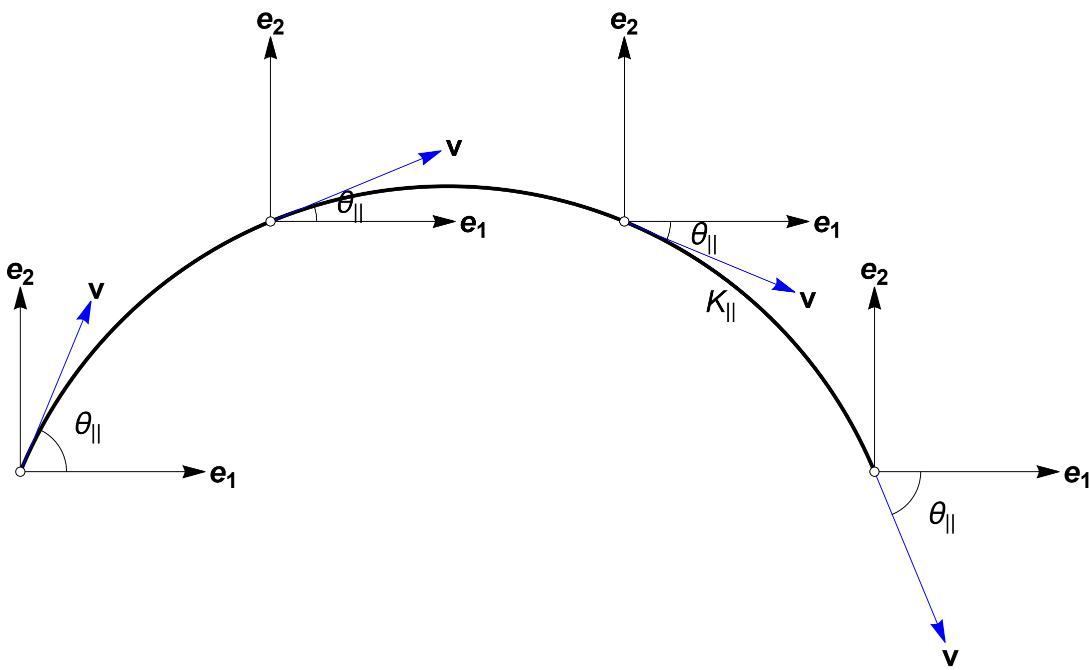

Figure 2. Surface curve K pressed flat to $K_{||}$
and traversed clockwise at unit speed

(iii) Draw a sketch illustrating $\boldsymbol{v}$, ($\boldsymbol{e_1}$, $\boldsymbol{e_2}$), *and* $\theta_{||}$ *at several points along K, after the narrow strip of peel surrounding K has been removed from S and pressed flat onto the plane.*

See Figure 2. K has been removed and pressed flat to become $K_{||}$, which, by sheer chance, happened to have a circular shape.

(iv) Deduce that the geodesic curvature of K *is indeed the ordinary curvature of the plane curve obtained by flattening the narrow strip surrounding K.*

The proposition can be deduced from $\boldsymbol{\kappa}_g = D_{\boldsymbol{v}}\, \boldsymbol{v} = \left[D_{\boldsymbol{v}}\, \theta_{||}\right] \boldsymbol{v}^{\perp}$. *The expression* $D_{\boldsymbol{v}}\, \theta_{||}$ represents change in an angle between the tangent and a parallel transported vector on successive tangent planes as p traverses a surface curve. When the curve is pressed onto a flat, Euclidean plane, the tangent planes are also pressed into and superimposed on the E plane. When the angular changes at each point are multiplied by $\boldsymbol{v}^{\perp}$, the result is a component orthogonal to velocity. It is the acceleration, which is equivalent to the “ordinary” curvature in the plane.

Comment: On a surface a “straight” line is a geodesic. So “geodesic curvature” actually measures departure of **v** from the direction of a geodesic. This can be a point of confusion. A *geodesic curve* has zero geodesic curvature, but *geodesic curvature* is not limited to geodesic curves. If a surface curve pressed or projected onto the plane shows measurable or visible positive or negative curvature κ *on the* *plane*, one can call that the intrinsic geodesic curvature.