Needham, VDGAF, Chapter 31, Exercise 3, pp. 334-335.
Palmer, December 19, 2023, 1:50 pm, PDT

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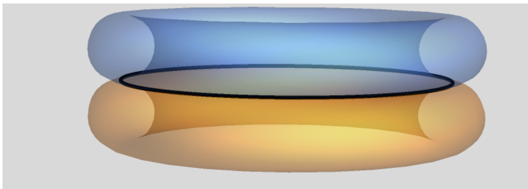

Figure 1. Torii touching along non-geodesic.

3. Geodesic Curvature along Touching Surfaces*. Suppose that two surfaces touch along a common curve C.*

(i) If C is a geodesic of one surface. explain geometrically why it must also be a geodesic of the other surface.

The explanation derives from the second paragraph of 22.2 and from [22.4], p. 235, which explains that on a geodesic, the original velocity is always parallel transported along itself. Hence, for a second surface touching along C, the velocity is the same vector and it is still transported along itself. Hence, it must also be a geodesic for the second surface.

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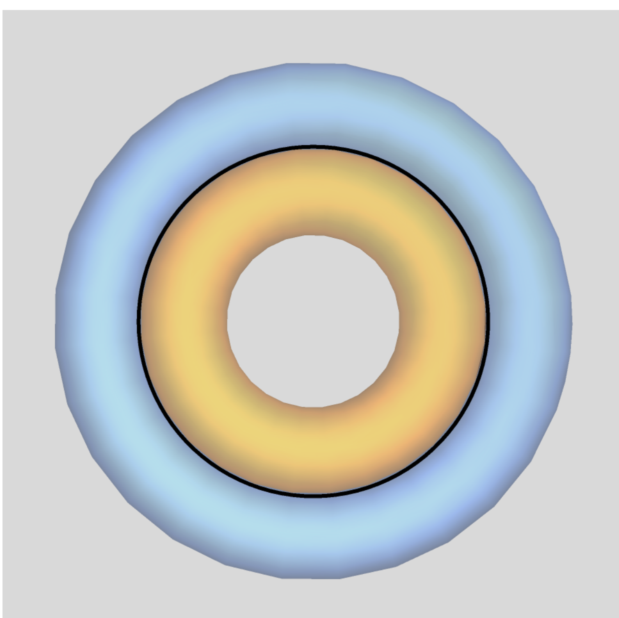

Figure 2. Two torii touching along geodesic on the outer-inner surface.

(ii) If C has geodesic curvature κ_g for one surface, explain geometrically why it must also have geometric curvature κ_g for the other surface.

If two surfaces touch at a point p, they share the same tangent plane at p, and this is true along the entire geodesic curve C (Figure 2). They also share $\mathcal{K}$, because $\mathcal{K}$ is a property of the shared curve. By [11.2], κ_g is a projection of $\mathcal{K}$ (i.e. acceleration) onto the tangent plane, so κ_g is the same for both surfaces that share a geodesic curve.

(iii) Deduce (i) as a special case of (ii).

(i) is the special case where the geodesic curvature of a surface $\kappa_g = 0$. Then, by (ii), it must also have geodesic curvature $\kappa_g = 0$ for the second surface. Then, by definition, it must be a geodesic.

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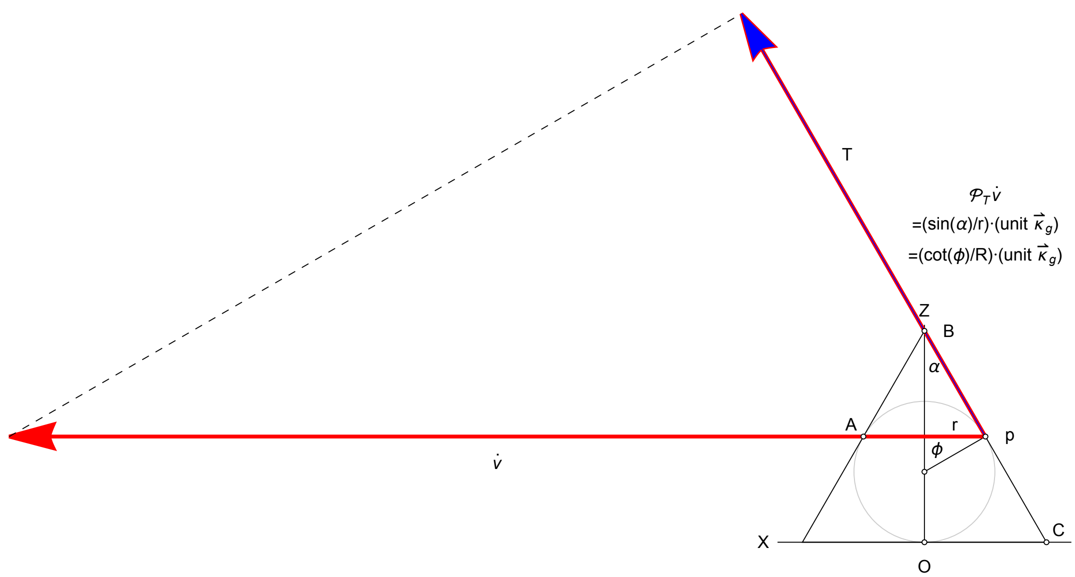

Figure 3. Circle of latitude (A to p) on surface of both sphere of radius R and cone of side length 1 (B to C), cross-section in XZ-plane.

(iv) Use the formulas for κ_g in the previous two exercises to confirm (ii) in the case that one surface is a sphere, and the other surface is a cone, and C is a circle of latitude on the sphere.

The formula for geodesic curvature of C on the sphere is $k_g = \cot(\phi)/R$. Figure 1 shows that $(\cot\phi)/R$ is equivalent to the projection of the acceleration onto the tangent plane T of S at p = {r, 0} on C. Since the acceleration $\dot{\boldsymbol{v}}$ is projected onto T along a shadow of a meridian and at the same time onto the generator of the cone that intersects C at p, because the generator lies in T, we have only to show that $\cot(\phi)/R = \sin(\alpha)/r$. Since the acceleration of p in the plane of C is the same for both surfaces touched by C, it projects onto the same vector in both cases. This is shown in Figure 3 by the purple color, which is created by the superposition of the blue projected κ_g of the cone onto the red projected κ_g of the sphere.