Needham, VDGAF, Chapter 31, Exercise 2, p. 334
Palmer, April 17 , 2024, 9:30, PDT

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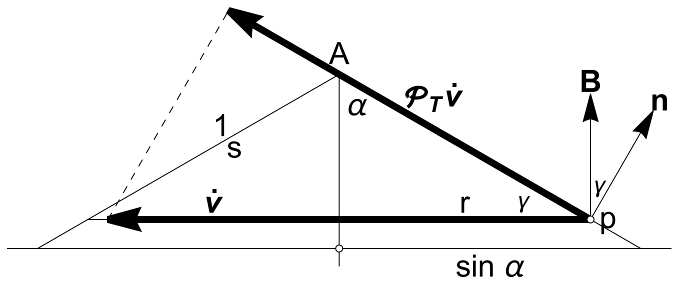

Figure 1. Cross section of cone with moving particle on circle

2. Geodesic Curvature on the Cone. *Reconsider the cone of semivertical angle α shown in [14.3], page 145. Suppose a particle travels at unit speed along a horizontal circle of radius r on this cone.*

(i) What is the magnitude and direction of the acceleration of the particle?

Note: For this and subsequent parts, the v arrow in Figure 1 indicates only direction, not magnitude. Let C be an expression for the circle with radius r, and let θ be the angle of rotation about the Z-axis as the particle traverses C at unit speed as in Exercise 1 of this chapter. The centre of rotation is constant $Z_C = \{0,\ 0,\ A - r\cos\alpha\}$. Since speed is distance over time, we introduce the parameter t, writing it explicitly in θ(t) or implicitly with superscript dots for derivatives as $\dot{C}$, $\dot{\theta}$, etc.

What is the magnitude?

$$s = r\,\theta(t)$$

$$v = \dot{s} = r\,\dot{\theta} = 1 \Longrightarrow \dot{\theta} = \frac{1}{r}$$

$$\boldsymbol{C} = (r\cos\theta,\ r\sin\theta,\ Z_C)$$

We find the velocity as the first derivative of **C**.

$$\boldsymbol{v} = \left(-r\sin\theta\,\dot{\theta},\ r\cos\theta\,\dot{\theta},\ 0\right)$$

$$= (-\sin\theta,\ \cos\theta,\ 0), \quad \text{by substituting for } \dot{\theta}$$

As in Exercise 1, we find the second derivative of **C**, which is the acceleration.

$$\dot{\boldsymbol{v}} = \left(-\cos\theta\,\dot{\theta},\; -\sin\theta\,\dot{\theta},\; 0\right)$$

$$= \left(-\frac{\cos\theta}{r},\; -\frac{\sin\theta}{r},\; 0\right), \quad \text{by substituting for } \dot{\theta}$$

Let X, Y, be the horizontal axes. The magnitude of acceleration is $\sqrt{X^2 + Y^2} = \frac{1}{r}$. The direction of acceleration of p is towards the Z axis as shown by the minus signs in $\dot{\boldsymbol{v}}$.

(ii) Sketch the projection of this acceleration onto the tangent plane of the cone, and deduce that the geodesic curvature is $\kappa_g = \frac{\sin\alpha}{r}$.

For the sketch, see $\boldsymbol{\mathcal{P}_T}\,\dot{\boldsymbol{v}}$ in Figure 1. The geodesic curvature is evidently the projection of the acceleration onto the tangent plane along a meridian of the cone, i.e. the projection in the XZ-plane along a geodesic generator. As p traverses the circle, this projection would also trace a circle at the tip of the $\boldsymbol{\kappa_g} = \boldsymbol{\mathcal{P}_T}\,\dot{\boldsymbol{v}}$ vector.

By [11.2], p. 116, γ is the angle between the normal to the surface and the binormal to the acceleration κ on the curve. Then, $\gamma = (\pi/2 - \alpha)$ in Figure 1. By (11.1), since $\kappa = 1/r$,

$$\kappa_g = \kappa \cos\gamma = \kappa\cos(\pi/2 - \alpha) = \kappa \sin\alpha = \frac{\sin\alpha}{r}$$

(iii) Let s be the distance within the cone (along a geodesic generator) from the vertex to the circle. Cut the cone along a generator and press it flat onto the plane, as in [14.3], page 145, so that the circle becomes an arc of a circle in the plane of radius s and therefor of curvature (1/s) within the plane. Verify that this curvature is the same as κ_g. ...

We are being asked to show that $\frac{1}{s} = \kappa_g = \frac{\sin\alpha}{r}$. In Figure 1, we can see that the outside distance along a geodesic generator equals 1. By symmetrical similar triangles $\frac{1}{s} = \frac{\sin\alpha}{r}$. Done.

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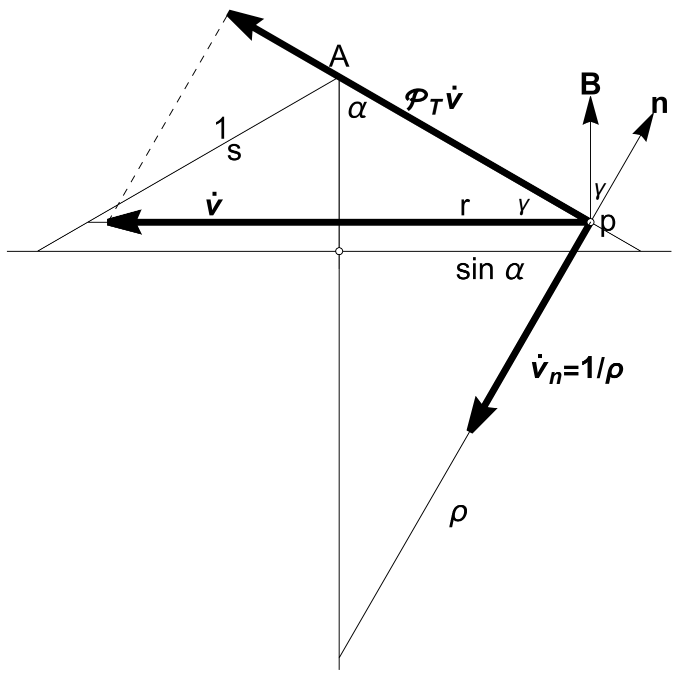

Figure 1. Cross section of cone
with moving particle on circle

(iv) Let ρ be the distance along the normal from the particle to the axis of symmetry. Show that the component of acceleration along the normal is $(1/\rho)$, and explain this geometrically.

The cone can be viewed as a surface of revolution. By analogy to [10.5], p. 113, $\kappa_n = \kappa_2 = -\frac{1}{\rho}$. The minus sign seems incorrect. The curvature along the normal is the acceleration along the normal, thus explaining the ratio $\frac{1}{\rho}$. Geometrically, the circle with curvature κ becomes an ellipse when projected onto a plane orthogonal to the meridian at p. The radius of curvature taken at p along the normal lengthens accordingly to become ρ and the curvature $1/\rho$ lessens proportionally.