Needham, VDGF, Chapter 31, Exercise 13, p. 339.
Palmer, July 16, 2024, 5 pm, PDT

13. Eclipses and Tides*. The tidal influence of the moon on the oceans is more than twice as great as the Sun's, despite the fact that the Sun's gravitational pull on the earth is about 200 times more powerful than the Moon's....*

(1) ...If the radii of the moon and the sun are r_m and r_s, and their distances from Earth are R_m and R_s, deduce that $(R_m / R_s) \approx (r_m / r_s)$.

Since solar eclipses occur, the angle θ subtended by the moon at the site of observation on the earth must be approximately equal to the angle subtended by the sun at that site. Then we can use the segments of circles centred on the earth and intersecting the Moon and Sun at their visible boundaries as approximations to $2\,r_m$ and $2\,r_w$. We can write:

$$2\,r_m \approx R_m\,\theta$$

$$2\,r_s \approx R_s\,\theta$$

It follows that

$$\theta \approx 2\,r_m / R_m \approx 2\,r_s / R_s,$$

from which it follows that

$$r_m / r_s \approx R_m / R_s$$

(ii) We have seen that the tidal force exerted on the oceans by a body of mass M at distance R from Earth is proportional to M / R^3. By appealing to (i), deduce that the ratio of the lunar to solar tidal forces is the ratio of the densities of the Moon to the Sun.

Let F be tidal force, ρ be density, M be mass, α be a constant, and V be volume with subscripts m and s for Moon and Sun. It is given that

$$F_m = \alpha \frac{M_m}{{R_m}^3} \qquad\qquad F_s = \alpha \frac{M_s}{{R_s}^3}$$

Write the ratio, substitute r_m and r_s for r_m and r_s by (i), divide by $(4/3)\pi$ on top and bottom to get V

$$\frac{F_m}{F_s} = \frac{\alpha \frac{M_m}{{R_m}^3}}{\alpha \frac{M_s}{{R_s}^3}} = \frac{\frac{M_m}{(4/3)\,\pi\, {r_m}^3}}{\frac{M_s}{(4/3)\,\pi\, {r_s}^3}} = \frac{\frac{M_m}{V_m}}{\frac{M_s}{V_s}} = \frac{\rho_m}{\rho_s}$$

This is what we wanted.

(iii) The average density of the moon is approximately 3300 kilograms per cubic meter, while that of the sun is approximately 1400 kilograms per cubic meter. Use (ii) to explain the strange opening fact.

By “strange opening fact” we assume Needham means *The tidal influence of the moon on the oceans is more than twice as great as the Sun’s, despite the fact that the Sun’s gravitational pull on the earth is about 200 times more powerful than the Moon’s.*

$$\frac{F_m}{F_s} = \frac{\rho_m}{\rho_s} = \frac{3300\ \text{kg}/m^3}{1400\ \text{kg}/m^3} = 2.35$$

The result using (ii) shows that *The tidal influence of the moon on the oceans is more than twice as great as the Sun’s.* Using Newton’s Inverse Square Law of Gravity, one can compare the gravitational pull of the Moon and the Sun on the Earth. Let G_m and G_s be the gravitational forces of the Moon and Sun. Let **G** be the gravitational constant. M and R are mass and distance from the Earth, as above.

$$\frac{G_m}{G_s} = \frac{\mathbf{G} M_m M_e}{{R_m}^2} \cdot \frac{{R_s}^2}{\mathbf{G} M_s M_e} = \frac{{R_s}^2}{{R_m}^2} \cdot \frac{M_m}{M_s} = \left(1.515 \times 10^5\right) \cdot \left(3.695 \times 10^{-8}\right) = 0.005597$$

And 1/0.005597 ≈ 179, meaning that the gravitational pull of the Sun on the Earth is approximately 179 times that of the Moon. This is not quite the 200 times used by Needham, but perhaps it is close enough given that the distance of the Sun from the Earth varies greatly with the seasons.

Linear distances appear in the denominators for the calculations of both tidal and gravitational forces. The radii used in calculating tidal forces are cubed. Distances from Earth to the Moon and Sun are squared. Longer distances such as the radius of the Sun and the distance to the Sun reduce tidal forces by the power of three and gravitational forces by the power of two. Hence, the Moon’s tidal force is surprisingly strong by comparison to the gravitational force of the Sun.

Relevant measurements:
Mass of Sun: 1.988×10^{30} km
Mass of Moon: $7.346\ \ 10^{22}$ kg
Radius of Sun: $6.957\ 10^5$ km
Radius of Moon: 1737 km
Distance to Sun: $149.6\ 10^6$ km
Distance to Moon: $384.4\ 10^3$ km