Needham, VDGF, Chapter 31, Exercise 12, p. 339.
Palmer, June 21, 2025, 9:15 am, PDT

12.

(i) Reconsider the derivation of the Riemann tensor in [29.8], but suppose for simplicity's sake that $[\boldsymbol{u},\ \boldsymbol{v}] = 0$*, so that the parallelogram closes up, in which case the curvature operator simplifies to*

$$\boldsymbol{R}(\boldsymbol{u},\ \boldsymbol{v}) = [\nabla_{\boldsymbol{u}},\ \nabla_{\boldsymbol{v}}].$$

Explain why the vector holonomy of the parallelogram is given by

$$-\delta\, \boldsymbol{w}_{||} = [exp(\delta u \nabla_{\boldsymbol{u}}),\ \ exp(\delta v \nabla_{\boldsymbol{v}})]\, \boldsymbol{w}.$$

It is given that

$$f(a\ +\ \delta \mathrm{x})\ =\ \exp\left[\delta \mathrm{x} \tfrac{d}{dx}\right] f\ |_{a} \qquad (1)$$

and by analogy that

$$\boldsymbol{w}(a + \boldsymbol{u}\delta \mathrm{u}) = \exp[\delta \mathrm{u} \nabla_{\boldsymbol{u}}]\, \boldsymbol{w}\ |_{a} \qquad (2)$$

$\boldsymbol{w}(a + \boldsymbol{u}\delta \mathrm{u})$ in (2) is the value of **w** at the point $(a + \boldsymbol{u}\delta \mathrm{u})$, which is very near to a. This also implies that

$$\boldsymbol{w}(a + \boldsymbol{u}\delta \mathrm{u}) \asymp \delta \mathrm{u} \nabla_{\boldsymbol{u}}\, \boldsymbol{w}(a) \qquad (3)$$

One can deduce from lines 1 to 5 of the introduction to this exercise that $exp(\delta u \nabla_{\boldsymbol{u}})$ can be rewritten as the Taylor's series:

$$\exp(\delta \mathrm{u} \nabla_{\boldsymbol{u}}) \equiv \left\{1 + \delta \mathrm{u} \nabla_{\boldsymbol{u}} + \tfrac{1}{2!}\, (\delta \mathrm{u})^2\, (\nabla_{\boldsymbol{u}})^2 + \tfrac{1}{3!}\, (\delta \mathrm{u})^3\, (\nabla_{\boldsymbol{u}})^3 + \ldots\right\} \qquad (4)$$

and there is a parallel series for $\exp(\delta \mathrm{v} \nabla_{\boldsymbol{v}})$. To help understand the operators, treat **w** as though it were $f\ =\ \{w_1,\, w_2,\, w_3\}$ and let $\nabla_{\boldsymbol{u}} = \left\{\frac{d}{du_i}\right\}$. Then, by (1),

$$\boldsymbol{w}(a + \boldsymbol{u}\delta \mathrm{u}) = \left\{\exp\left[\delta \mathrm{u} \tfrac{d}{du_i}\right] w_i\right\}\ |_{a}$$

This analysis is not needed in the proof.

The a in (2) is abstract. Letters "a", "b", "p", and "o" will henceforth refer to actual points on the parallelogram such as that in [29.7]. We are also given that

$$-\delta \boldsymbol{w}_{||} = \delta \mathrm{u} \delta \mathrm{v} \mathcal{R}(\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{w} = [\delta \mathrm{u} \nabla_{\boldsymbol{u}}, \delta \mathrm{v} \nabla_{\boldsymbol{v}}]\, \boldsymbol{w} = \{\delta \mathrm{u} \nabla_{\boldsymbol{u}}\, \delta \mathrm{v} \nabla_{\boldsymbol{v}} - \delta \mathrm{v} \nabla_{\boldsymbol{v}}\, \delta \mathrm{u} \nabla_{\boldsymbol{u}}\}\, \boldsymbol{w}. \qquad (5)$$

This gives the difference of two instances of $\boldsymbol{w}_{||}$ after anticlockwise traversal on a parallelogram such as that in [29.7], first from position o *via* a to b, where we find $\boldsymbol{w}_{||}(b)$, then continuing on *via* p back to o, where we find $\boldsymbol{w}_{||}(o)$. We omit point q because the parallelogram closes in this exercise. The ordering of the two derivatives in each second derivative summand requires some derivation.

Using the $\boldsymbol{w}(a + \boldsymbol{u}\delta \mathrm{u})$ notation, we can show $\boldsymbol{w}_{||}$ on the traversal of the parallelogram.

First two sides: $\boldsymbol{w}_{||\mathrm{oab}}$ is the result of the traversal $\left(\boldsymbol{w}_{||}(o) \to \boldsymbol{w}_{||}(o + \boldsymbol{u}\delta \mathrm{u}) \to \boldsymbol{w}_{||}(a + \boldsymbol{v}\delta \mathrm{v})\right)$. The result is named $\boldsymbol{w}_{||}(b)$.

Second two sides: $\boldsymbol{w}_{||\mathrm{bpo}}$ is the result of the traversal $\left(\boldsymbol{w}_{||}(b) \to \boldsymbol{w}_{||}(b - \boldsymbol{u}\delta \mathrm{u}) \to \boldsymbol{w}_{||}(p - \boldsymbol{v}\delta \mathrm{v})\right)$. The result is named $\boldsymbol{w}_{||}(o)$ to distinguish it from $\boldsymbol{w}(o)$.

Note that for the derivation beginning on p. 289, Needham has apparently chosen a regular vector field. That is, the fiducial vector is constant across the parallelogram, so $\boldsymbol{w}(o)$, $\boldsymbol{w}(a)$, $\boldsymbol{w}(b)$ and $\boldsymbol{w}(p)$ are all instances of the same fiducial vector anchored at different points on the traversal. Consequently, they can all be written **w**, as in line 4 of the derivation. During a traversal, the parallel-transported vector rotates away from the fiducial vector field represented by $\boldsymbol{w}$ and is given a "||" subscript, as in $\boldsymbol{w}_{||}(b)$. Notice that $\delta \boldsymbol{w}_{||}(b)$ refers to a **difference** from the fiducial vector field at b. On the basis of the the derivation on p. 289, one can write

$$\delta \boldsymbol{w}_{||}(a) \asymp \boldsymbol{w}_{||}(o + \boldsymbol{u}\delta \mathrm{u}) - \boldsymbol{w} \asymp \delta \mathrm{u} \nabla_{\boldsymbol{u}}\, \boldsymbol{w}(o)$$
$$\delta \boldsymbol{w}_{||}(b) \asymp \boldsymbol{w}_{||}(a + \boldsymbol{v}\delta \mathrm{v}) - \boldsymbol{w} \asymp \delta \mathrm{v} \nabla_{\boldsymbol{v}}\, \boldsymbol{w}(a)$$
$$\delta \boldsymbol{w}_{||}(p) \asymp \boldsymbol{w}_{||}(b - \boldsymbol{u}\delta \mathrm{u}) - \boldsymbol{w} \asymp -\delta \mathrm{u} \nabla_{\boldsymbol{u}}\, \boldsymbol{w}(p)$$
$$\delta \boldsymbol{w}_{||}(o) \asymp \boldsymbol{w}_{||}(p - \boldsymbol{v}\delta \mathrm{v}) - \boldsymbol{w} \asymp -\delta \mathrm{v} \nabla_{\boldsymbol{v}}\, \boldsymbol{w}(o)$$

As on p. 289, these sum to

$$-\delta \boldsymbol{w}_{||} = -\left[\delta \boldsymbol{w}_{||}(a) + \delta \boldsymbol{w}_{||}(b) + \delta \boldsymbol{w}_{||}(p) + \delta \boldsymbol{w}_{||}(o)\right]$$

$$\asymp \delta \mathrm{u} \nabla_{\boldsymbol{u}}\, \boldsymbol{w}(o) + \delta \mathrm{v} \nabla_{\boldsymbol{v}}\, \boldsymbol{w}(a) - \delta \mathrm{u} \nabla_{\boldsymbol{u}}\, \boldsymbol{w}(p) - \delta \mathrm{v} \nabla_{\boldsymbol{v}}\, \boldsymbol{w}(o)$$

$$\asymp \delta \mathrm{u} \delta \mathrm{v}\, \{\nabla_{\boldsymbol{u}}\, \nabla_{\boldsymbol{v}}\, \boldsymbol{w}\} - \delta \mathrm{u} \delta \mathrm{v}\, \{\nabla_{\boldsymbol{v}}\, \nabla_{\boldsymbol{u}}\, \boldsymbol{w}\} \quad \text{(by last line, p. 289; intermediate step omitted)}$$

$$= \{\delta \mathrm{u} \nabla_{\boldsymbol{u}}\} \circ \{\delta \mathrm{v} \nabla_{\boldsymbol{v}}\}\, \boldsymbol{w} - \{\delta \mathrm{v} \nabla_{\boldsymbol{v}}\} \circ \{\delta \mathrm{u} \nabla_{\boldsymbol{u}}\}\, \boldsymbol{w}$$

$$\asymp \{\exp[\delta \mathrm{u} \nabla_{\boldsymbol{u}}] \times \exp[\delta \mathrm{v} \nabla_{\boldsymbol{v}}] - \exp[\delta \mathrm{v} \nabla_{\boldsymbol{v}}] \times \exp[\delta \mathrm{u} \nabla_{\boldsymbol{u}}]\}\, \boldsymbol{w}$$

[By (2), (3), and (4) above, substituting products of exponential operator for $\{\delta u \triangledown_{\boldsymbol{u}}\} \circ \{\delta v \triangledown_{\boldsymbol{v}}\}$.]

Because derivatives factorize and obey the Index Law $D^m . D^n\, u = D^{m+n} . u$ (Paggio, H.T.H., *Elementary Treatise*...,1949, p. 31), they must also follow the rules of the Riemann bracket notation. That is, one can write $D^m . E^n\, u$ and $[D, E]\, u = D.Eu - E.Du$, where D and E are derivatives, such as $\frac{d}{dx}$ or $\triangledown_{\boldsymbol{u}}$. The exponential operator is actually a derivative, or more precisely, a sum of derivatives, because it is a Taylor's series applied to the gradient. Then one can write

$$-\delta \boldsymbol{w}_{\|} = [\exp(\delta u \triangledown_{\boldsymbol{u}}),\ \exp(\delta v \triangledown_{\boldsymbol{v}})]\, \boldsymbol{w}$$

This is what we wanted.

(ii) Deduce that

$$-\delta\, \boldsymbol{w}_{\|} = \delta u \delta v\, \boldsymbol{\mathcal{R}}(\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{w} + \textit{(third-order error)}$$

where the "third-order error" is made up of terms involving $(\delta u)^p (\delta v)^q$*, where* $(p+q) \geq 3$.

In part (i) we found that

$$-\delta \boldsymbol{w} \asymp \exp(\delta u \triangledown_{\boldsymbol{u}}) \cdot \exp(\delta v \triangledown_{\boldsymbol{v}})\, \boldsymbol{w} - \exp(\delta v \triangledown_{\boldsymbol{v}}) \cdot \exp(\delta u \triangledown_{\boldsymbol{u}})\, \boldsymbol{w} \qquad (5)$$

and

$$\exp(\delta u \triangledown_{\boldsymbol{u}}) \equiv \left\{1 + \delta u \triangledown_{\boldsymbol{u}} + \frac{1}{2!} (\delta u)^2 (\triangledown_{\boldsymbol{u}})^2 + \frac{1}{3!} (\delta u)^3 (\triangledown_{\boldsymbol{u}})^3 + \ldots\right\}$$

and there is a parallel series for $\exp(\delta v \triangledown_{\boldsymbol{v}})$. Then by composing the terms in the series, the first term $\exp(\delta u \triangledown_{\boldsymbol{u}}) \cdot \exp(\delta v \triangledown_{\boldsymbol{v}})$ in (5) becomes

$$\begin{aligned}
&1 \cdot \left\{1 + \delta v \triangledown_{\boldsymbol{v}} + \frac{1}{2!} (\delta v)^2 (\triangledown_{v})^2 + \ldots\right\} \\
&+\delta u \triangledown_{\boldsymbol{u}} \circ \left\{1 + \delta v \triangledown_{\boldsymbol{v}} + \frac{1}{2!} (\delta v)^2 (\triangledown_{v})^2 + \ldots\right\} \\
&+\frac{1}{2!} (\delta u)^2 (\triangledown_{\boldsymbol{u}})^2 \circ \left\{1 + \delta v \triangledown_{\boldsymbol{v}} + \frac{1}{2!} (\delta v)^2 (\triangledown_{v})^2 + \ldots\right\} \\
&\ldots \\
&= \left\{1 + \delta v \triangledown_{\boldsymbol{v}} + \frac{1}{2!} (\delta v)^2 (\triangledown_{v})^2 + \ldots\right\} + \left\{\delta u \triangledown_{\boldsymbol{u}} + \delta u \triangledown_{\boldsymbol{u}} \circ \delta v \triangledown_{\boldsymbol{v}} + \delta u \triangledown_{\boldsymbol{u}} \circ \frac{1}{2!} (\delta v)^2 (\triangledown_{v})^2 + \ldots\right\} \\
&+\left\{\frac{1}{2!} (\delta u)^2 (\triangledown_{\boldsymbol{u}})^2 + \frac{1}{2!} (\delta u)^2 (\triangledown_{\boldsymbol{u}})^2 \diamond \delta v \triangledown_{\boldsymbol{v}} + \frac{1}{2!} (\delta u)^2 (\triangledown_{\boldsymbol{u}})^2 \circ \frac{1}{2!} (\delta v)^2 (\triangledown_{v})^2 + \ldots\right\}
\end{aligned}$$

Write just the terms where $p + q \leq 2$ and subtract the second term $\exp(\delta v \triangledown_{\boldsymbol{v}}) \cdot \exp(\delta u \triangledown_{\boldsymbol{u}})$ in (3).

$$= \{1 + \delta v \triangledown_{\boldsymbol{v}} + \delta u \triangledown_{\boldsymbol{u}} + \delta u \triangledown_{\boldsymbol{u}} \circ \delta v \triangledown_{\boldsymbol{v}} + (\text{terms with } p + q \geqslant 3)\}$$
$$- \{1 + \delta v \triangledown_{\boldsymbol{v}} + \delta u \triangledown_{\boldsymbol{u}} + \delta v \triangledown_{\boldsymbol{v}} \circ \delta u \triangledown_{\boldsymbol{u}} + (\text{terms with } p + q \geqslant 3)\}$$
$$= \{\delta u \triangledown_{\boldsymbol{u}} \circ \delta v \triangledown_{\boldsymbol{v}} - \delta v \triangledown_{\boldsymbol{v}} \circ \delta u \triangledown_{\boldsymbol{u}} + (\text{terms with } p + q \geqslant 3)\}$$

because in general, $\delta u \triangledown_{\boldsymbol{u}} \circ \delta v \triangledown_{\boldsymbol{v}} \neq \delta v \triangledown_{\boldsymbol{v}} \circ \delta u \triangledown_{\boldsymbol{u}}$.

This shows that

$$\{\exp(\delta u \triangledown_{\boldsymbol{u}}) \cdot \exp(\delta v \triangledown_{\boldsymbol{v}}) - \exp(\delta v \triangledown_{\boldsymbol{v}}) \cdot \exp(\delta u \triangledown_{\boldsymbol{u}})\} \boldsymbol{w}$$

$$= \{\delta u \triangledown_{\boldsymbol{u}} \circ \delta v \triangledown_{\boldsymbol{v}} - \delta v \triangledown_{\boldsymbol{v}} \circ \delta u \triangledown_{\boldsymbol{u}} + (\text{terms with } p + q \geqslant 3)\} \boldsymbol{w}$$

$$= \{\boldsymbol{[}\delta u \triangledown_{\boldsymbol{u}}, \ \delta v \triangledown_{\boldsymbol{v}}\boldsymbol{]} \ + \ (\text{terms with } p + q \geqslant 3)\} \boldsymbol{w}$$

From (29.5) and the first box on p. 288, one can write:

$$\boldsymbol{[}\delta u \triangledown_{\boldsymbol{u}}, \ \delta v \triangledown_{\boldsymbol{v}}\boldsymbol{]} \, \boldsymbol{w} \boldsymbol{=} (\delta u \delta v) \, \boldsymbol{\mathcal{R}}(\boldsymbol{u}, \ \boldsymbol{v}) \, \boldsymbol{w}$$

It follows that

$$-\delta \boldsymbol{w} \boldsymbol{=} (\delta u \delta v) \, \boldsymbol{\mathcal{R}}(\boldsymbol{u}, \ \boldsymbol{v}) \, \boldsymbol{w} + (\text{third} - \text{order error})$$