Needham, Chapter 31, Exercise 11, p. 338.
G. Palmer, Jun 30, 2025, 4 pm PDT.

11. Counting the components of the Riemann Tensor. ...

(i) Given that the components R_{ijkl} of the Riemann Tensor in an n-manifold are antisymmetric in ij and kl, deduce that there are $P = \frac{1}{2}n(n-1)$ nontrivial ways of choosing pairs ij, and likewise P ways of choosing pairs kl.

The first pair of orthogonal bases, ij, has three components for each of i and j, so in sum they have six components and in combination they have $n^2 = 9$ ways of choosing pairs. But three of those ways pair identical bases, so we can subtract n identical or "trivial" pairings from the total, leaving six. Then, the first antisymmetry reduces the number of nontrivial ways of choosing ij by half.

$$P = \left(n^2 - n\right)/2 = (9-3)/2 = 3 = (1/2)\times 3\times(3-1) = (1/2)\,n\,(n-1)$$

But what makes the identical base pairs "trivial" and subject to subtraction? *Is it because* $\mathcal{R}(u,\ u) = 0$, as in (29.7)? Then to argue the same for jk, one would need to apply the pair-pair symmetry of (29.16). It can also be worked out using (29.13).

(ii) Given the symmetry (29.16) under interchange of the first and second pair of indices, $R_{ijkl} = R_{klij}$ (see previous exercise [i.e. 10]) deduce that if we [also] *take these pair symmetries into account, there are $\frac{1}{2}P(P+1)$ independent ways of choosing ijkl.*

Given the results of part (i), the pairs ij and kl in combination account for P^2 ways of choosing ijkl. Following Vasco's practice of letting $I \equiv$ ij, and $J \equiv$ kl, the symmetry klij = ijkl provides an additional way of pairing the two slots i and l. They can now be paired as li or the antisymmetry -il, resulting in P additional ways of arranging ijkl. Hence, one can write

$$W = \left(\tfrac{1}{2}\right)P(P+1) = \tfrac{1}{2}\left(P^2 + P\right).$$

where W is all the ways of choosing pairs of ijkl given only the first two antisymmetries and the IJ symmetry.

But we are not supposed to know about the antisymmetry of li, because it is not proven until part (iii) of this exercise. Therefore, it is better to follow Vasco's solution, which begins like mine with P^2 total ways of combining I and J pairs. But Vasco notes that IJ can be regarded as a single pair like ij or kl, except that—unlike ij or kl where equal base vector pairs like 22 or 33 are trivial—IJ has no trivial components (ways of combining its vectors). An equal IJ pair would be 1212, 1313, 2323, ... These are not trivial. Vasco substitutes P for n in the equation $P = \frac{1}{2}n(n-1)$ and then adds the P nontrivial pairs for the ways of pairing of IJ, giving the equation

$$W = \left(\tfrac{1}{2}\right) P(P+1) = \tfrac{1}{2} P(P-1) + P$$

which is what we wanted.

(iii) Defining $B_{ijkl} \equiv = R_{ijkl} + R_{iklj} + R_{iljk}$, *as in the previous exercise [i.e. 10], the Bianchi Symmetry (29.15)* [p. 295] *states that* $B_{ijkl} = 0$ *(see previous exercise).* [Note that this should be $B_{\text{ijkl}} \equiv R_{\text{ijkl}} + R_{\text{jkil}} + R_{\text{kijl}}$. The equation given can not be used to verify the asymmetries. However, my second approach does verify them as well as this $B_{\text{ijkl}} = 0$.] *Verify that the pair symmetries now ensure that* B_{ijkl} *is totally* ***antisymmetric*** [emphasis added] *on all four indices, and that the constraint* $B_{ijkl} = 0$ *is therefore trivially satisfied unless all four indices are distinct. [Note: The latter condition would require at least a 4-manifold.]*

By "totally antisymmetric" I think Needham must mean that any swap of the order of any two base vectors must result in equality of absolute value and opposite sign. Swaps between pairs of Riemann vectors conserve sign. Using $B_{\text{ijkl}} \equiv R_{\text{ijkl}} + R_{\text{jkil}} + R_{\text{kijl}}$, we are to show that

$B_{\text{jikl}} = -B_{\text{ijkl}}$ (j, i) swap
$B_{\text{ijlk}} = -B_{\text{ijkl}}$ (l, k) swap
$B_{\text{kjil}} = -B_{\text{ijkl}}$ (k, i) swap
$B_{\text{ikjl}} = -B_{\text{ijkl}}$ (k, j) swap
$B_{\text{ilkj}} = -B_{\text{ijkl}}$ (l, j) swap
$B_{\text{ljki}} = -B_{\text{ijkl}}$ (l, i) swap

Vasco in his answer has shown that a single method works for four of these asymmetries. Write out the Bianchi identity **for the swap** with indices on the Riemann tensors in the revised (rotated) order appropriate to the swap. RS = Riemann symmetry; RAS1 = first Riemann antisymmetry ; RAS2 = second Riemann antisymmetry. The asymmetries and the symmetry are defined in section 29.5.7, pp. 294-295.

Keep the Bianchi identity in view as a template.
$B_{\text{ijkl}} = R_{\text{ijkl}} + R_{\text{jkil}} + R_{\text{kijl}}$

(1) $B_{\text{jikl}} = R_{\text{jikl}} + R_{\text{ikjl}} + R_{\text{kjil}}$ (j, i) swap

$B_{\text{kjil}} = -R_{\text{ijkl}} - R_{\text{kijl}} - R_{\text{jkil}}$ (by RAS1)

$= -\left(R_{\text{ijkl}} + R_{\text{kijl}} + R_{\text{jkil}}\right) = -B_{\text{ijkl}}$

(2) $B_{\text{jikl}} = R_{\text{jikl}} + R_{\text{ikjl}} + R_{\text{kjil}}$ (j, i) swap

$B_{\text{kjil}} = -R_{\text{ijkl}} - R_{\text{kijl}} - R_{\text{jkil}}$ (by RAS1)

$= -(R_{\text{ijkl}} + R_{\text{kijl}} + R_{\text{jkil}}) = -B_{\text{ijkl}}$

(3) $B_{\text{kjil}} = R_{\text{kjil}} + R_{\text{jikl}} + R_{\text{ikjl}}$ (k, i) swap

$B_{\text{kjil}} = -R_{\text{jkil}} - R_{\text{ijkl}} - R_{\text{kijl}}$ (by RAS1)

$= -(R_{\text{jkil}} + R_{\text{ijkl}} + R_{\text{kijl}}) = -B_{\text{ijkl}}$

(4) $B_{\text{ikjl}} = R_{\text{ikjl}} + R_{\text{kjil}} + R_{\text{jikl}}$ (k, j) swap

$= -(R_{\text{kijl}} + R_{\text{jkil}} + R_{\text{ijkl}}) = -B_{\text{ijkl}}$

(5) $B_{\text{ilkj}} = R_{\text{ilkj}} + R_{\text{lkij}} + R_{\text{kilj}}$ (l, j) swap

Apply RS to 2nd term.

$= R_{\text{likj}} + R_{\text{ijlk}} + R_{\text{kilj}}$

Apply RAS2 to 2nd and 3rd term.

$= R_{\text{likj}} - R_{\text{ijkl}} - R_{\text{kilj}} = -B_{\text{ijkl}}$

Apply RS, RAS1, and RAS2 to first term.

$= R_{\text{jkil}} - R_{\text{ijkl}} - R_{\text{kilj}} = -B_{\text{ijkl}}$

(6) $B_{\text{ljki}} = R_{\text{ljki}} + R_{\text{jkli}} + R_{\text{klji}}$ (l, i) swap

Apply RS to terms 1 and 3.

$R_{\text{kilj}} + R_{\text{jkli}} + R_{\text{jikl}}$

Apply RAS2 to term 1 and RAS1 to term 3.

$-R_{\text{kijl}} + R_{\text{jkli}} - R_{\text{ijkl}}$

Apply RAS2 to term 2.

$= -R_{\text{kilj}} - R_{\text{jkil}} - R_{\text{ijkl}} = -B_{\text{ijkl}}$

[Verify] *that the constraint $B_{ijkl} = 0$ is therefore trivially satisfied unless all four indices are distinct. [Note: The latter condition would require at least a 4-manifold.]*

Applying asymmetries to the 2nd and 3rd terms,

$$B_{\text{ijkl}} = 0 = R_{\text{ijkl}} + R_{\text{jkil}} + R_{\text{kijl}}$$

$$B_{\text{ijkl}} = 0 = R_{\text{ijkl}} + R_{\text{ijkl}} + R_{\text{ijkl}}$$

$= 3\, R_{\text{ijkl}}$

So $B_{\text{ijkl}} = 0$ is trivially satisfied if $R_{\text{ijkl}} = 0$.

(iv) If n < 4, the Bianchi symmetry therefore does not impose any new constraints. Use (ii) to deduce that if n = 2 the Riemann tensor has one component (the Gaussian curvature), and if n = 3 it has six components.

Apply $\frac{1}{2}\, P(P+1)$, where $P = \frac{1}{2}\, n(n-1)$, to $n = 2$.

$P = \frac{1}{2} \times 2 \times (2-1) = 1$

$\frac{1}{2}\, P(P+1) = \frac{1}{2} \times 1 \times (1+1) = 1$ component

Apply to $n = 3$.

$P = \frac{1}{2} \times 3 \times (3-1) = 3$

$\frac{1}{2} \times 3 \times (3+1) = \frac{1}{2} \times 12 = 6$ components

(v) If n ⩾ 4, deduce that the number of additional constraints resulting from the Bianchi symmetry is equal to the number of ways of choosing four objects from n objects.

Assume that the n objects mentioned in the exercise are n basis vectors indexed by ijkl where n ⩾ 4. Then the problem is to determine how many fewer components (more constraints) are permitted by the number of ways of choosing four objects from n objects. Judging from part (vi), this number is

C (constraints) = $\frac{n!}{(n-4)!\, 4!}$

$n = 4 \qquad C = 1$
$n = 5 \qquad C = 5$
$n = 6 \qquad C = 15$

(vi) Deduce that the number of independent components of the Riemann tensor is

$\frac{1}{2}\, P(P+1) - \frac{n!}{(n-4)!\, 4!}$,

in which the second term correctly disappears if n < 4. [Because you can’t choose n distinct objects from fewer than n objects.]

It is clear that there are at least P * P ways of combining two base pairs with two other base pairs and that the symmetry of the pairs reduces the combinations by half, so we arrive at $\frac{1}{2}P(P)$. But why is the first term $\frac{1}{2}P(P+1)$? But given that it is correct as is, the number of additional constraints introduced by larger n decreases the possible combinations by

$$\frac{n!}{(n-4)!\,4!}$$

The following table checks to assure that

$$\frac{1}{2}P(P+1)-\frac{n!}{(n-4)!\,4!}=\frac{1}{12}n^2(n^2-1)$$

Table 1. Checking $\frac{1}{2}P(P+1)-\frac{n!}{(n-4)!\,4!}$ against (29.1).

$P=\frac{1}{2}n\,(n-1)$

C_{ijkl}(independent ways of choosing ijkl) $=\frac{1}{2}P(P+1)$

C (constraints) $=\frac{n!}{(n-4)!\,4!}$

(29.1) (indep. components of RT) $=\frac{1}{12}n^2(n^2-1)$

n	P	C_{ijkl}	C	$C_{ijkl}-C$	(29.1)
2	1	1	0	1	1
3	3	6	0	6	6
4	6	21	1	20	20
5	10	55	5	50	50
6	15	120	15	105	105

(vii) Verify that this does indeed yield formula (29.1): The number of independent components of the Riemann tensor is

$$\frac{1}{12}n^2(n^2-1).$$

$$P=\frac{1}{2}n(n-1)$$

$$\frac{1}{2}P(P+1)-\frac{n!}{(n-4)!\,4!}=\frac{1}{2}\times\left(\frac{1}{2}n(n-1)\right)\times\left(\frac{1}{2}n(n-1)+1\right)-\frac{n!}{(n-4)!\,4!}$$

$$=\frac{1}{4}n(n-1)\left(\frac{1}{2}n(n-1)+1\right)-\frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!\,4!}$$

$$=\frac{1}{4}n(n-1)\left(\frac{1}{2}n(n-1)+1\right)-\frac{n(n-1)(n-2)(n-3)}{4!}$$

$$= \frac{1}{8}\, n(n-1)[(n(n-1)+2)] - \frac{n(n-1)\,(n-2)\,(n-3)}{4\cdot 3\cdot 2}$$

$$= \frac{1}{8}\, n(n-1)[(n(n-1)+2)] - \frac{n(n-1)}{8}\, \frac{(n-2)\,(n-3)}{3}$$

$$= \frac{1}{8}\, n(n-1)\left[(n(n-1)+2) - \frac{(n-2)\,(n-3)}{3}\right]$$

$$= \frac{1}{8}\, n(n-1)\left(\frac{3\,n^2-3\,n+6}{3} - \frac{n^2-5\,n+6}{3}\right)$$

$$= \frac{1}{8}\, n(n-1)\left(\frac{2\,n^2+2\,n}{3}\right) = \frac{1}{4}\, n(n-1)\left(\frac{n^2+n}{3}\right)$$

$$= \frac{1}{12}\, n(n-1)\, n(n+1) = \frac{1}{12}\, n^2\left(n^2-1\right)$$

The credit for this derivation goes to Vasco.