Needham, VDGF, Chapter 31, Exercise 10, p. 337-338.
Palmer, May 7, 2024, 10:50 am, PDT

10. Two Symmetries of the Riemann Tensor.

(i) Let us confirm the First (Algebraic) Bianchi Identity, (29.15), which is also called the ***Bianchi Symmetry****:*

$$\boldsymbol{\mathcal{R}(u,\ v)\,w + \mathcal{R}(v,\ w)\,u + \mathcal{R}(w,\ u)\,v} = 0 \quad \Longleftrightarrow \quad R_{ijkm} + R_{jkim} + R_{kijm} = 0$$

Confirming: Introduce a fourth vector and take the inner product with the LHS

$$\{\boldsymbol{\mathcal{R}(u,v)w+\mathcal{R}(v,w)u+\mathcal{R}(w,u)v\}} \cdot \mathbf{x} = 0 \cdot \mathbf{x} = 0 \qquad (1)$$

Following the double frame on p. 293 and the equation below it, rewrite equation (1).

$$\left\{R_{ijk}{}^{l}\, u^i\, v^j\, w^k\, \boldsymbol{e_l} + R_{jki}{}^{l}\, v^j\, w^k\, u^i\, \boldsymbol{e_l} + R_{kij}{}^{l}\, w^k\, u^i\, v^j\, \boldsymbol{e_l}\right\} \cdot \boldsymbol{e_m} = 0$$

$$\left\{R_{ijk}{}^{l}\, u^i\, v^j\, w^k\, \boldsymbol{e_l} \cdot \boldsymbol{e_m} + R_{jki}{}^{l}\, v^j\, w^k\, u^i\, \boldsymbol{e_l} \cdot \boldsymbol{e_m} + R_{kij}{}^{l}\, w^k\, u^i\, v^j\, \boldsymbol{e_l} \cdot \boldsymbol{e_m}\right\} = 0$$

And by (29.9) and (29.10), assuming that $\boldsymbol{e_l} \neq 0$ and $\boldsymbol{e_m} \neq 0$, and $u^i\, v^j\, w^k = v^j\, w^k\, u^i = w^k\, u^i\, v^j$, divide both sides by $u^i\, v^j\, w^k$, giving

$$R_{ijkm} + R_{jkim} + R_{kijm} = 0$$

confirming the first equation of Bianchi symmetry.

To prove the general result it actually suffices to prove it in the case that all three vector fields are coordinate fields, in which case their commutators all vanish. Prove that in this case,

$$\boldsymbol{\mathcal{R}(u,\ v)\,w + \mathcal{R}(v,\ w)\,u + \mathcal{R}(w,\ u)\,v = \nabla_u[v,\ w] + \nabla_v[w,\ u] + \nabla_w[u,\ v] = 0}$$

Since it is shown in the first single frame on p. 288 that the square brackets represent the commutators for three different sets of coordinate vector fields (**v, w**), (**w, u**) and (**u, v**), the three terms with square brackets left of 0 are all zero by construction. But we can also calculate the result to show that the sum between the equal signs is 0.

$$\boldsymbol{\mathcal{R}(u,\ v)\,w = [\nabla_u,\ \nabla_v]\,w} = (\nabla_u \nabla_v - \nabla_v \nabla_u)\,\boldsymbol{w} = \nabla_u \nabla_v\, \boldsymbol{w} - \nabla_v \nabla_u\, \boldsymbol{w} \qquad \text{(a)}$$
$$\boldsymbol{\mathcal{R}(v,\ w)\,u = [\nabla_v,\ \nabla_w]\,u} = (\nabla_v \nabla_w - \nabla_w \nabla_v)\,\boldsymbol{u} = \nabla_v \nabla_w\, \boldsymbol{u} - \nabla_w \nabla_v\, \boldsymbol{u} \qquad \text{(b)}$$
$$\boldsymbol{\mathcal{R}(w,\ u)\,v = [\nabla_w,\ \nabla_u]\,v} = (\nabla_w \nabla_u - \nabla_u \nabla_w)\,\boldsymbol{v} = \nabla_w \nabla_u\, \boldsymbol{v} - \nabla_u \nabla_w\, \boldsymbol{v} \qquad \text{(c)}$$

$$\boldsymbol{\mathcal{R}(u,\ v)\,w + \mathcal{R}(v,\ w)\,u + \mathcal{R}(w,\ u)\,v} = \nabla_u \nabla_v\, \boldsymbol{w} - \nabla_v \nabla_u\, \boldsymbol{w} + \nabla_v \nabla_w\, \boldsymbol{u} - \nabla_w \nabla_v\, \boldsymbol{u} + \nabla_w \nabla_u\, \boldsymbol{v} - \nabla_u \nabla_w\, \boldsymbol{v}$$ (See p.

290)

$= (\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} \boldsymbol{w} - \nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{w}} \boldsymbol{v}) + (\nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{w}} \boldsymbol{u} - \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}} \boldsymbol{w}) + (\nabla_{\boldsymbol{w}} \nabla_{\boldsymbol{u}} \boldsymbol{v} - \nabla_{\boldsymbol{w}} \nabla_{\boldsymbol{v}} \boldsymbol{u})$ by rearranging

$= \nabla_{\boldsymbol{u}}[\boldsymbol{v}, \boldsymbol{w}] + \nabla_{\boldsymbol{v}}[\boldsymbol{w}, \boldsymbol{u}] + \nabla_{\boldsymbol{w}}[\boldsymbol{u}, \boldsymbol{v}]$ by definition of **[u,v]** and linearity of **$\mathcal{R}$, p. 288**.

Since the commutators are 0 because the parallelograms close, and the order of the coordinates of second derivatives does not affect the absolute value of the final result, $\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} \boldsymbol{w} = \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}} \boldsymbol{w}$. Then the sum of the RHS of (a)+(b)+(c) equals zero.

Out[◦]=

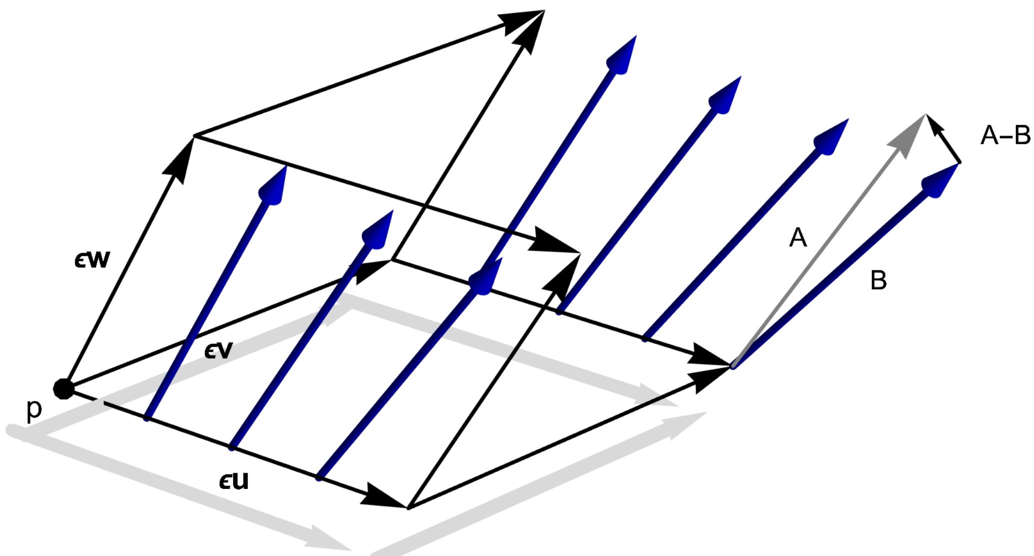

Figure 1. Holonomy on polyhedral box

Out[∘]=

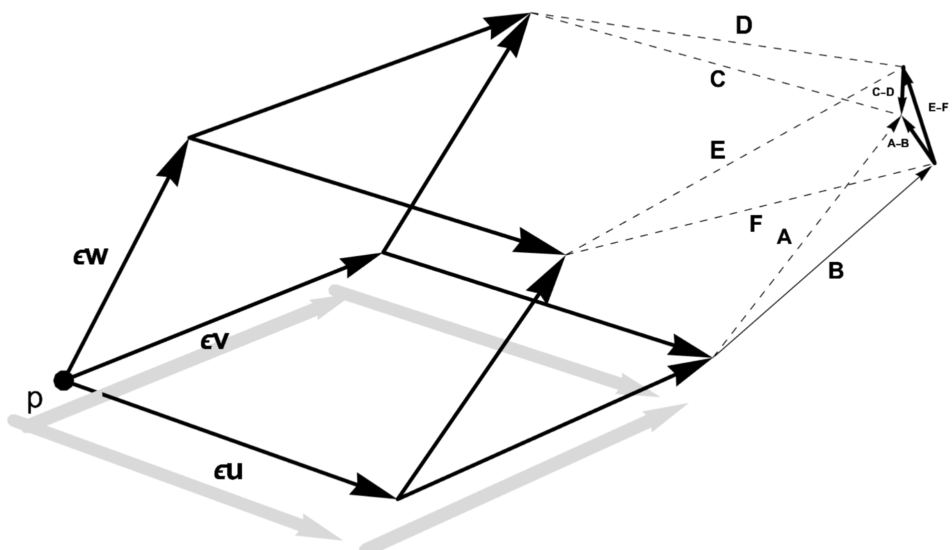

Figure 2. Closure on polyhedral box in 3–manifold with curvature

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(ii) Instructions are given for drawing a parallelogram box based on $\epsilon\boldsymbol{u}$, $\epsilon\boldsymbol{v}$, and $\epsilon\boldsymbol{w}$. From the instructions, it is difficult to see where the holonomy arises. Apparently the sides of the floor are not necessarily geodesics. Figure 1 illustrates a possible A and B and the possibility that the holonomy of $\epsilon\boldsymbol{w}$ along $\epsilon\boldsymbol{u}$ and $\epsilon\boldsymbol{v}$ originates in a downward curving surface (not shown in Figure 2) that bends along with increasing $\epsilon\boldsymbol{u}$. The curve somehow influences the direction of $\epsilon\boldsymbol{w}$ under transport despite the closure of the parallelogram on $\epsilon\boldsymbol{u}$ and $\epsilon\boldsymbol{v}$. We reason that all three vectors must be parallel transported simultaneously.

In Figure 2, I attempt to depict the final result, as described by Needham in the continuation of part (ii) on p. 338. I don’t understand exactly how the holonomy accomplishes closure of three triangular faces with sides {A, B, A-B}, {C, D, C-D}, {E, F, E-F} by means of the fourth face { A-B, C-D, E-F}.

In [29.6], the problem was to establish the ultimate dependence of curvature on <u>area</u> via the equation $\mathcal{K}(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{w}) \asymp \frac{\delta\Theta}{\delta\mathcal{A}} \asymp \left| \frac{\delta\boldsymbol{w}}{\delta\mathcal{A}} \right| \asymp |\, \boldsymbol{\mathcal{R}}(\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{w} \,|$. The box made of parallelograms, introduces <u>volume</u>, which the problem equation appears to reflect in the term ϵ^3. The linearity and independence of the components of the Riemann curvature formula enable moving the coefficient ϵ of the three vectors from inside the formula $\mathcal{R}(\epsilon\mathbf{u},\epsilon\mathbf{v};\epsilon\mathbf{w})$ to a single ϵ^3 representing volume, as in

$$A - B \asymp \mathcal{R}(\epsilon \boldsymbol{u},\ \epsilon \boldsymbol{v};\ \epsilon \boldsymbol{w}) = \epsilon^3\, \mathcal{R}(\boldsymbol{u},\ \boldsymbol{v};\ \boldsymbol{w})$$

where, following p. 287, ¶ 3, representing the edges as vectors,

$$\mid \mathcal{R}(\boldsymbol{u},\ \boldsymbol{v};\ \boldsymbol{w}) \mid\ = \left| \frac{\delta \boldsymbol{w}_{\parallel}}{\delta \mathcal{V}} \right| = \left| \frac{\boldsymbol{A}-\boldsymbol{B}}{\delta \mathcal{V}} \right| = \left| \frac{\boldsymbol{A}-\boldsymbol{B}}{\epsilon^3} \right|$$

Perhaps one could then add all the curvatures of the closed box,

$$\mid \mathcal{R}(\boldsymbol{u},\ \boldsymbol{v};\ \boldsymbol{w}) + \mathcal{R}(\boldsymbol{v},\ \boldsymbol{w};\ \boldsymbol{u}) + \mathcal{R}(\boldsymbol{w},\ \boldsymbol{u};\ \boldsymbol{v}) \mid\ = \left| \frac{(\boldsymbol{A}-\boldsymbol{B})+(\boldsymbol{C}-\boldsymbol{D})+(\boldsymbol{E}-\boldsymbol{F})}{\epsilon^3} \right| = \left| \frac{\delta\boldsymbol{u}+\delta\boldsymbol{v}+\delta\boldsymbol{w}}{\epsilon^3} \right|,$$

but according to the Bianchi Symmetry (29.15), this sum vanishes. What is the geometric meaning of that vanishing? If one treats (**A-B**), (**C-D**), and (**E-F**) as vectors, then they sum to zero. Since their sum is a zero vector, it has absolute value of zero, which is what we see in the Bianchi Symmetry. Hence, the geometry appears to illustrate the Bianchi Symmetry.

(iii) Let us confirm that the Riemann tensor is symmetric under the interchange of the first and second pairs of vectors (29.16):

$$[\mathcal{R}\,(\boldsymbol{u},\ \boldsymbol{v})\,\boldsymbol{x}] \bullet \boldsymbol{y} = [\mathcal{R}\,(\boldsymbol{x},\ \boldsymbol{y})\,\boldsymbol{u}] \bullet \boldsymbol{v} \iff R_{ijkl} = R_{klij}$$

In order to let the indices themselves signify their corresponding vectors, subscripted u, v, x, y is used in preference to subscripted i, j, k, l.

The four Bianchi Algebraic identities needed for the identity of the proof.

$$B_{\text{uvxy}} = R_{\text{uvxy}} + R_{\text{vxuy}} + R_{\text{xuvy}}$$
$$B_{\text{vxyu}} = R_{\text{vxyu}} + R_{\text{xyvu}} + R_{\text{yvxu}}$$
$$B_{\text{xyuv}} = R_{\text{xyuv}} + R_{\text{yuxv}} + R_{\text{uxyv}}$$
$$B_{\text{yuvx}} = R_{\text{yuvx}} + R_{\text{uvyx}} + R_{\text{vyux}}$$

By the Riemann identities $R_{\text{jikm}} = -R_{\text{ijkm}}$ (29.7) and $R_{\text{ijmk}} = -R_{\text{ijkm}}$ (29.13), we can write

$R_{\text{uvxy}} = -R_{\text{vuxy}}$
$R_{\text{vxuy}} = -R_{\text{vxyu}}$
$R_{\text{xuvy}} = R_{\text{uxyv}}$ (two swaps makes a plus)
$R_{\text{xyvu}} = -R_{\text{xyuv}}$
$R_{\text{yvxu}} = R_{\text{vyux}}$

Then, by cancelling like terms,

$$B_{uvxy} + B_{vxyu} = B_{xyuv} + B_{yuvx} \implies R_{uvyx} + R_{xyuv} = 0$$

$$\implies R_{xyuv} = -(-R_{uvxy}) \implies R_{xyuv} = R_{uvxy} \text{ (by (29.13))}$$

This *confirms that the Riemann tensor is symmetric under the interchange of the first and second pairs of vectors (29.16)*. The following Mathematica program performs the cancellations. They are not difficult to do manually, but it is tedious to write them out.

```
Clear["Global`*"]

(* The Algebraic Bianchi identities *)
buvxy = ruvxy + rvxuy + rxuvy;
bvxyu = rvxyu + rxyvu + ryvxu;
bxyuv = rxyuv + ryuxv + ruxyv;
byuvx = ryuvx + ruvyx + rvyux;

(* Riemann identities *)
ruvxy = -ruvyx;
rvxuy = -rvxyu;
rxuvy = ruxyv;
rxyvu = -rxyuv;
ryvxu = rvyux;
ryuxv = -ryuvx;

Simplify[buvxy + bvxyu == bxyuv + byuvx]
```

Out[ ]=
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ruvyx + rxyuv == 0
```