Needham, VDGAF, Chapter 31, Exercise 1, p. 334
Palmer, April 17 , 2024, 9:15 am PDT

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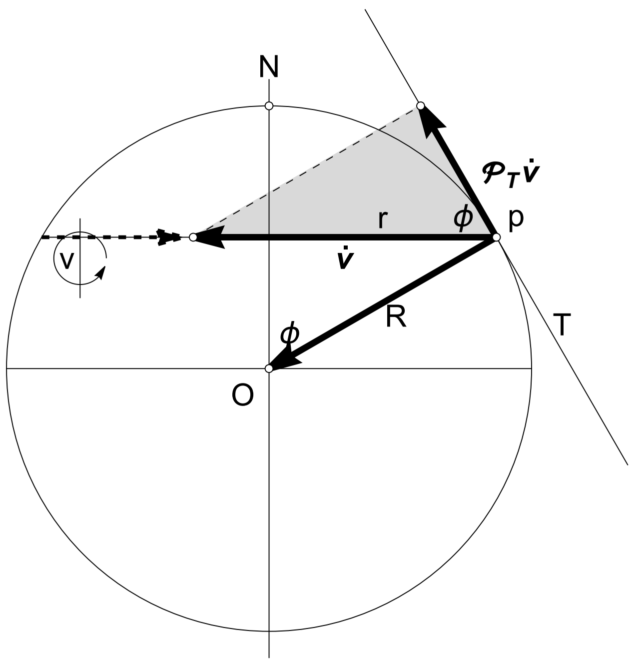

Figure 1. Sphere with particle p moving at unit speed v on surface; cross–section through XZ–plane

1. Geodesic Curvature on the Sphere. On the sphere of radius R, suppose a particle travels at unit speed along the circle at latitude ϕ.

(i) What is the magnitude and direction of the acceleration of the particle?

Referring to Figure 1, let α be an expression for the circle with radius r at constant latitude ϕ measured as the angle at the origin away from the Z-axis, and let θ be the angle of rotation about the Z-axis as the particle traverses α at unit speed as in [4.12][b], p. 48. The centre of rotation is constant $Z_\alpha = \{0,\ 0,\ R\cos\phi\}$. Since speed is distance over time, we introduce the parameter t, writing it explicitly in θ(t) or implicitly with superscript dots for derivatives as $\dot{\alpha}$, $\dot{\theta}$, etc. By construction, $r = R\sin\phi$.

What is the magnitude?

$$s = r\,\theta(t)$$

$$v = \dot{s} = r\,\dot{\theta} = 1 \Longrightarrow \dot{\theta} = \frac{1}{r}$$

$$\boldsymbol{\alpha} = (r\cos\theta,\ r\sin\theta,\ Z_\alpha)$$

$$\dot{\boldsymbol{\alpha}} = \left(-r\sin\theta\,\dot{\theta},\ r\cos\theta\ \dot{\theta},\ 0\right) \qquad [= \mathbf{v}]$$

The acceleration is $\ddot{\alpha}$, but differentiating $\dot{\alpha}$ would give an expression with both first and second derivatives, but we know $\theta = \frac{1}{r}$, which is a constant because ϕ is a constant, so we substitute for $\dot{\theta}$ in $\dot{\alpha}$ and obtain

$$\ddot{\boldsymbol{\alpha}} = \left(-\cos\theta\,\dot{\theta},\ -\sin\theta\,\dot{\theta},\ 0\right) \qquad [= \dot{\boldsymbol{v}}]$$

and substitute again for $\dot{\theta}$ to obtain

$$\ddot{\boldsymbol{\alpha}} = \left(-\frac{\cos\theta}{r},\ -\frac{\sin\theta}{r},\ 0\right)$$

Then the magnitude of the acceleration is

$$|\ \ddot{\boldsymbol{\alpha}}\ | = \frac{1}{r}$$

Since r is necessarily less than R by construction, if R = 1, the magnitude of the acceleration will exceed r, as in Figure 1. The length of v is not to scale, but the length of $\dot{\boldsymbol{v}}$ is.

What is the direction? The negative signs on both vector components show that the acceleration vector $\ddot{\boldsymbol{\alpha}}$ points towards the center of C, and in the case illustrated in Figure 1, passes through it.

(ii) Sketch the projection of this acceleration onto the tangent plane of the sphere, and deduce that the geodesic curvature is $\kappa_g = \frac{\cot\phi}{R}$.

For the sketch, see Figure 1. The geodesic curvature is the projection $\mathcal{P}_T(\dot{\boldsymbol{v}})$ of the acceleration onto the tangent plane along the shadow of a meridian with $\theta = 0$. The angle at p between r and $\mathcal{P}_T(\dot{\boldsymbol{v}})$ is ϕ

$$\mathcal{P}_T(\dot{\boldsymbol{v}}) = \dot{\boldsymbol{v}} \bullet \boldsymbol{T}$$

We wish to use the equation $\kappa_g = \kappa\cos\gamma$ from (11.2) and [11.2] with $\gamma = \phi$. We take the geodesic curvature κ_g to be the projection of the acceleration onto the tangent plane in the plane of a meridian. The curvature of the circle α on the surface of the sphere is $\kappa = \frac{1}{r} = \frac{1}{R\sin\phi}$.

$$\kappa_g = \kappa\cos\phi = \frac{1}{R\sin\phi}\cos\phi = \frac{\cot\phi}{R}$$

(iii) Verify that this formula for κ_g yields the geometrically correct answer as $\phi \to 0$ and as $\phi \to (\pi/2)$.

As $\phi \to 0$, the circle of traversal approaches the north pole and $\kappa_g = \text{Cot}(\phi)/R$ goes to 0. The geodesic acceleration vector ultimately becomes indistinguishable from the acceleration vector. The normal acceleration vector approaches the vertical. As κ_g goes to 0, the radius of geodesic curvature goes to infinity.

As $\phi \to (\pi/2)$, the circle of traversal approaches the equator, and $\kappa_g = \text{Cot}(\phi)/R$ goes to infinity and the normal acceleration vector becomes horizontal and directed towards the centre. The projection of acceleration onto the tangent plane to the sphere vanishes at $(\pi/2)$.

(iv) Show that the component of the acceleration directed towards the centre of the sphere has magnitude (1/R), independent of ϕ. Explain this geometrically.

In this construction, the component of acceleration directed towards the centre of the sphere is always a radius of length R from p to the origin, so its magnitude will ever be (1/R).

It is explained by Meusnier's Theorem (11.4), p. 117, which holds that $\kappa_n(T)$ is independent of γ, which is the angle of the plane of the osculating curve with the tangent plane at p. In this case, the osculating curve is the circle at latitude ϕ.

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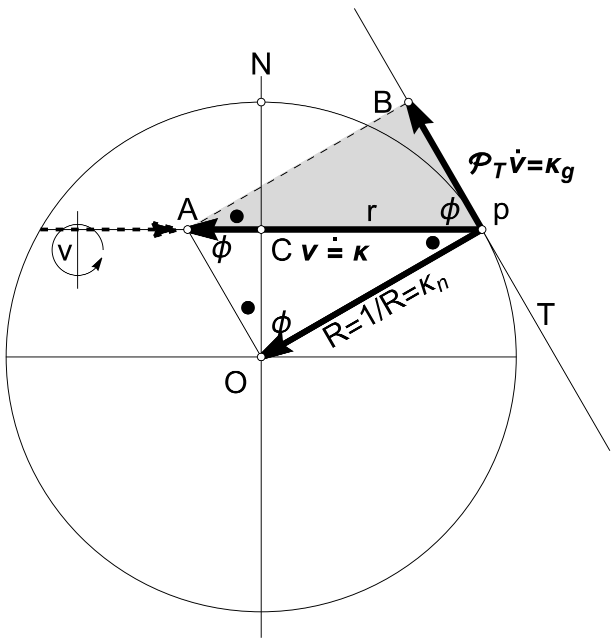

Figure 2. Acceleration rectangle
with unit speed particle v
on surface of unit sphere;
cross–section through XZ–plane

Additional observations

Based on similar triangles A-B-p and C-p-O with three angles in common, we can draw a line from the origin at a right angle to the p-ray, which necessarily meets $\dot{v}$ at the tip, because the two triangles O-A-p and A-B-p share a side and two angles, one of which is a right angle, thus completing a rectangle.]