Needham, Chapter 30, Notes

G. Palmer, July 3, 2025

P. 313. *Substituting this* $\left[\ddot{r} = f(r)\right]$ *in place of (30.1) in the above analysis, we find* ***[exercise]***,

$$\delta\ddot{\mathcal{V}} = 0 \implies \frac{df}{dr} + \frac{2f}{r} = 0 \implies f(r) \propto \frac{1}{r^2}$$

$$\ddot{r} = -\frac{\mathrm{GM}}{r^2} \implies f = -\frac{\mathrm{GM}}{r^2} \qquad (30.1)$$

$$\mathcal{K}_+ = \frac{\mathrm{GM}}{r^3} = -\frac{f}{r}, \qquad \mathcal{K}_- = -\frac{2\,\mathrm{GM}}{r^3} = \frac{2f}{r}$$

$$\delta\ddot{\mathcal{V}} = -\frac{4\pi}{3}\,(2\,\mathcal{K}_+ + \mathcal{K}_-)\,(\delta\mathrm{r})^3 = -\frac{4\pi}{3}\left(2\left(-\frac{f}{r}\right) + \frac{2f}{r}\right)(\delta\mathrm{r})^3 = -\frac{4\pi}{3}\,(0)\,(\delta\mathrm{r})^3 = 0$$

At this point, we have shown that substituting $\ddot{r} = f(r)$ in place of (30.1) gives $\delta\ddot{\mathcal{V}} = 0$, but we have not obtained $\frac{df}{dr} + \frac{2f}{r} = 0$. Note that, initially, $\xi = \delta\mathrm{r} = \Xi$.

$$\ddot{\Xi} = \left[\partial_r\,\ddot{r}\right]\delta\mathrm{r} = [\partial_r f]\,\delta\mathrm{r} = \frac{\mathrm{df}}{\mathrm{dr}}\,\delta\mathrm{r} \qquad \text{[from p. 312, above line -1]}$$

$$\ddot{\xi} = -\mathcal{K}_+\,\xi = \frac{f}{r}\,\xi = \frac{f}{r}\,\delta\mathrm{r} \qquad (30.2)$$

From the equation above (30.4) we have

$$\delta\ddot{\mathcal{V}} = -\frac{4\pi}{3}\left(2\,\ddot{\xi} + \ddot{\Xi}\right)(\delta\mathrm{r})^3 = -\frac{4\pi}{3}\left(\frac{2f}{r}\,\delta\mathrm{r} + \frac{\mathrm{df}}{\mathrm{dr}}\,\delta\mathrm{r}\right)(\delta\mathrm{r})^2 = -\frac{4\pi}{3}\left(\frac{\mathrm{df}}{\mathrm{dr}} + \frac{2f}{r}\right)(\delta\mathrm{r})^3 = 0$$

Hence,

$$\frac{df}{dr} + \frac{2f}{r} = 0 \implies \frac{df}{dr} = -\frac{2f}{r} \implies \frac{df}{f} = -2\,\frac{\mathrm{dr}}{r}$$

$$\int\frac{df}{f} = -2\int\frac{\mathrm{dr}}{r} \implies \log f = -2\log r + c \implies f = \frac{e^c}{r^2} \implies f(r) \propto \frac{1}{r^2}$$

Pp. 317-318. §30.6 Einstein's Vacuum Field Equation in Geometrical Form

Out[ ]=

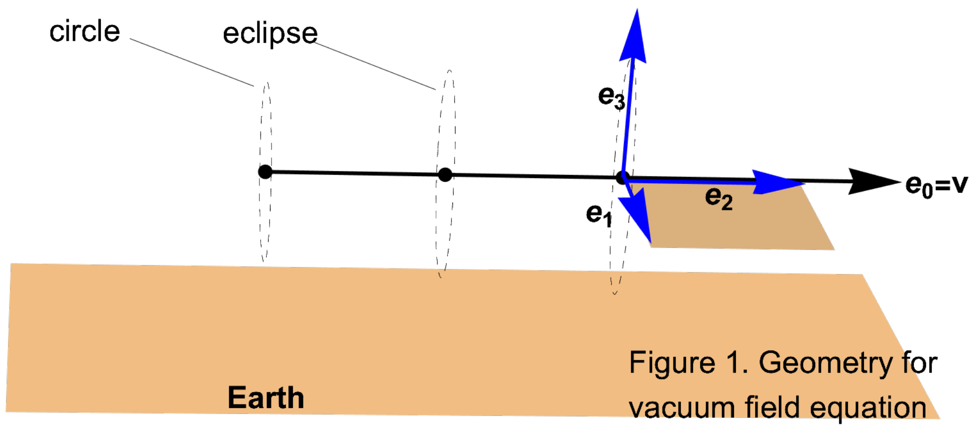

Figure 1. Geometry for vacuum field equation

Out[ ]=

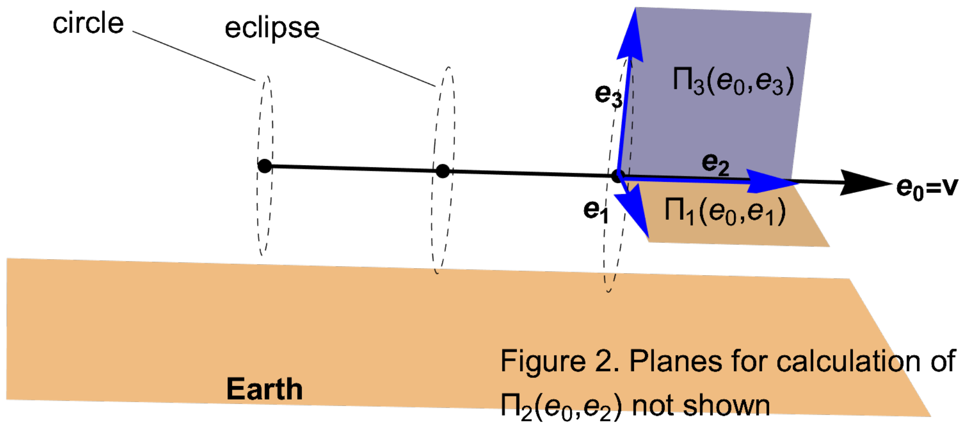

Figure 2. Planes for calculation of $\mathcal{K}_{mean}$
$\Pi_2(e_0,e_2)$ not shown

Figure 1 is based on section 30.6 and figure [30.4] rotated by $-\pi/2$ about the Y axis in order to show the surface of the Earth as horizontal. I deduce from the text that $\boldsymbol{e_0}$, the time axis, is orthogonal to $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, and $\boldsymbol{e_3}$ even though $\boldsymbol{e_0}$ is aligned with $\boldsymbol{e_2}$ in the diagram. The arrow labeled $\boldsymbol{e_0} = \boldsymbol{v}$ is the "world line."

v	the 4-velocity
$\boldsymbol{e_0}$	same as **v**, "a point along the ... time axis"
$\boldsymbol{e_1}$	basis vector in plane parallel to earth's surface
$\boldsymbol{e_2}$	basis vector in plane parallel to earth's surface
$\boldsymbol{e_3}$	points radially away from the Earth, in direction of the gravitational field

Figure 2 is based on Figure 1 and the calculation of $\mathcal{K}_{\text{mean}}$ from the sectional curvatures of planes containing the world line on p. 318.

P. 327. *But (30.16) is true if* ***v*** *is an arbitrarily timelike vector. So, using the same trick as before* [See p. 319, top], *writing* $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$ *and appealing to the symmetry of* ***T****, we deduce [****exercise****] that*

$$\boldsymbol{R}icci = 4\,\pi G \boldsymbol{T} \quad \Longleftrightarrow \quad R_{ik} = 4\,\pi G T_{ik}.$$

T is a scalar (§30.9, p. 326, ¶ 3). T_{ik} must also be a scalar calculated using only the first and third basis vectors in ijkl of the Riemann Tensor. The symmetry $\boldsymbol{T}(\boldsymbol{w},\ \boldsymbol{v}) = \boldsymbol{T}(\boldsymbol{v},\ \boldsymbol{w}) \iff T_{\text{ki}} = T_{\text{ik}}$ is given (30.15). It is also given that

$$\boldsymbol{R}\text{icci}(\boldsymbol{v}, \boldsymbol{v}) = 4\,\pi G \rho_{\text{total}} = 4\,\pi \boldsymbol{T}(\boldsymbol{v}, \boldsymbol{v}) \qquad (30.16)$$

$$R_{00} = 4\,\pi G \rho_{\text{total}} \qquad (30.17), \text{ using } \boldsymbol{v} = \boldsymbol{e}_0$$

Writing $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$, as on p. 319,

$$0 = \boldsymbol{R}\text{icci}(\boldsymbol{v}, \boldsymbol{v})$$

$$= \boldsymbol{R}\text{icci}(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y})$$

$$= \boldsymbol{R}\text{icci}(\boldsymbol{x},\ \boldsymbol{x}) + \boldsymbol{R}\text{icci}(\boldsymbol{x},\ \boldsymbol{y}) + \boldsymbol{R}\text{icci}(\boldsymbol{y},\ \boldsymbol{x}) + \boldsymbol{R}\text{icci}(\boldsymbol{y},\ \boldsymbol{y})$$

$$= 0 + \boldsymbol{R}\text{icci}(\boldsymbol{x},\ \boldsymbol{y}) + \boldsymbol{R}\text{icci}(\boldsymbol{y},\ \boldsymbol{x}) + 0$$

$$= 2\mathbf{R}\text{icci}(\mathbf{x}, \mathbf{y})$$

And by the same trick,

$$0 = \boldsymbol{T}(\boldsymbol{v}, \boldsymbol{v})$$

$= \boldsymbol{T}(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y})$

$= 2\,\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y})$

In (39.15), the energy-momentum tensor $\boldsymbol{T}(\boldsymbol{v}, \boldsymbol{w})$ corresponds to T_{ik}, which appears to be the energy-momentum tensor applied to the first and third basis vectors. Then it must be the case that

$$\boldsymbol{R}icci(\boldsymbol{v}, \boldsymbol{w}) = 4\,\pi G\boldsymbol{T}(\boldsymbol{v}, \boldsymbol{w}) \iff R_{ik} = 4\,\pi G T_{ik}$$

or

$$\boldsymbol{R}icci = 4\,\pi G\boldsymbol{T} \iff R_{ik} = 4\,\pi G T_{ik},$$

where the first vector is understood to be **v** and the second vector is understood to be the vector undergoing parallel-transport. This is true for both tensors—**R**icci and **T**.

§30.9, p. 326-328. The Einstein Field Equation (with Matter) in Geometrical Form (30.24)

The following is my attempt to acquire some intuition for understanding (30.24) by posing two problems: the acceleration of volume of a cubic meter of a gel and the acceleration of volume of the Galaxy (assuming it initially static). I may have some assumptions wrong, so I welcome comments.

On pp. 313-314, ¶ -1, Needham wrote: *But suppose we remove you from the interior of the sphere, and instead fill it with extremely dense matter of density ρ. Then the sphere of particles of radius $\xi = \Xi$ will be accelerated inward towards the centre, just as if there were a point mass $\rho\delta\mathcal{V}$ at the centre, pulling them in with acceleration*

$$\ddot{\xi} = -\frac{G\rho\delta\mathcal{V}}{\xi^2}.$$

Where does this equation come from? Since $\xi = r$ and $M = \rho\delta\mathcal{V}$,

$$\ddot{r} \asymp -\frac{\text{GM}}{r^2} \implies \ddot{\xi} \asymp -\frac{\text{GM}}{r^3}\,\xi \implies \ddot{\xi} = -\frac{\text{G}\rho\delta\mathcal{V}}{\xi^2} \qquad \text{(from (30.1) and first equation p. 312)}$$

Needham continued: *Now let us compute the acceleration of the volume in this new circumstance. Since now $\delta\mathcal{V} = \frac{4\pi}{3}\,\xi^3$,*

$$\delta\dot{\mathcal{V}} = 4\,\pi\xi^2\,\dot{\xi} \implies \delta\ddot{\mathcal{V}} = 8\,\pi\xi\,(\dot{\xi})^2 + 4\,\pi\xi^2\,\ddot{\xi} = 8\,\pi\xi\,(0)^2 + 4\,\pi\left(-\frac{G\rho\delta\mathcal{V}}{\xi^2}\right).$$

In the box on p. 314, he wrote:

***Geometrical Significance of the Inverse-Square Attraction**:*
Consider a sphere of volume $\delta\mathcal{V}$ that is filled with matter of density ρ, and suppose that just above its surface are tiny test particles, released from rest. Instantly they begin to accelerate towards the centre, and the inverse-square law causes the volume they enclose to implode with an acceleration governed by this geometrical law:

$$\delta\ddot{\mathcal{V}} = -4\,\pi G \rho \delta\mathcal{V}\,.$$

This is an abbreviated version of the field equation (30.24):

$$\delta\ddot{\mathcal{V}} = -4\,\pi\mathrm{G}(\rho_{\mathrm{total}} + P_1 + P_2 + P_3)\,\delta\mathcal{V} = -4\,\pi\mathrm{GT}\delta\mathcal{V}$$

Where G is the gravitational constant, ρ is density of matter and/or energy, and $\delta\mathcal{V}$ is volume. T, P, and Ricci will not be used. T is the stress-energy tensor, P is pressure on a slice of a manifold, and **R**icci refers to the Ricci tensor.

$$\boldsymbol{R}\mathrm{icci}(v,\ v)\,\delta\mathcal{V} = -\delta\ddot{\mathcal{V}} = 4\,\pi\mathrm{G}\rho\delta\mathcal{V} \qquad (30.14)$$

For reference:

$G \simeq 6.7\,x\,10^{-11}\,m^3/(\mathrm{kg}\cdot s^2)$ (gravitational constant, m^3)

$G \simeq 6.7\,x\,10^{-20}\,\mathrm{km}^3/(\mathrm{kg}\cdot s^2)$ (gravitational constant, km^3)

Shrinkage of Cubic Meter of Atoms

Suppose one has 1 m^3 sphere of atoms weighing 1 kg. What will be the acceleration of shrinkage of volume in the absence of the gravity of the earth or any other neighboring body?

$$T = \rho = 1\,\mathrm{kg}/m^3$$
$$\delta\mathrm{V} = 1\,m^3$$

Then,

$$\delta\ddot{\mathcal{V}} = -4\,\pi\mathrm{G}\cdot\rho\cdot\delta\mathcal{V} = -4\,\pi\left(6.7\,x\,10^{-11}\,m^3/(\mathrm{kg}\cdot s^2)\,(1\,\mathrm{kg}/m^3)\times 1\,m^3 = -\frac{8.42\times10^{-10}\,m^3}{s^2}\right.$$

The acceleration of volume is tiny, as one might expect in a cubic meter of gas or gel. It is roughly one thousandth of a cubic centimeter per second squared.

Acceleration of Shrinkage of the Galaxy

Next calculate the acceleration of the volume of the Galaxy. Assume that the Galaxy is initially static, lacking rotation.

$m^3 == 10^6\ \text{cm}^3$
$\text{ly}^3 == 10^{48}\ m^3$
$\mathcal{V}_{\text{galaxy}} \simeq 10^{12}\ \text{ly}^3$ (Google AI)
$\rho = 0.1\ \text{atom}/\text{cm}^3$
$h = 1.67\ \text{x}10^{-27}\ \text{kg}$ (mass of hydrogen atom)
$\rho_H = 1.67\ \text{x}10^{-28}\ \text{kg}/\text{cm}^3$ (density of hydrogen atoms in galaxy)
$\rho_{\text{galaxy\_km}} = 1.67\ \text{x}10^{-22}\ \text{kg}/m^3$ (density of hydrogen atoms in galaxy)
$\rho_{\text{galaxy\_ly}} = \left(1.67\ \text{x}10^{-22}\ \text{kg}/m^3\right) \times \left(10^{48}\ m^3/\text{ly}^3\right) = 1.67\ \text{x}10^{26}\ \text{kg}/\text{ly}^3$ (ρ hydrogen atoms in G)
$\delta\mathcal{V} = \mathcal{V}_{\text{galaxy}} = 10^{12}\ \text{ly}^3$
$G = 6.7\ x\ 10^{-11}\ m^3/\left(\text{kg}\ s^2\right)$ (Gravitational constant)
$G_{\text{ly}} = G\ x\ 10^{-48}\ \text{ly}^3/m^3 = 6.7\ x\ 10^{-59}\ \text{ly}^3/\left(\text{kg}\ s^2\right)$ (Gravitational constant in light years per kg s^2)

$$\begin{aligned}\delta\ddot{\mathcal{V}} &= -4\ \pi \text{G}_{\text{ly}} \cdot \rho_{\text{galaxy\_ly}} \cdot \delta\mathcal{V} \\ &= -4\ \pi\left(6.7\ x\ 10^{-59}\ \text{ly}^3/\text{kg}\ s^2\right)\left(1.67\ \text{x}10^{26}\ \text{kg}/\text{ly}^3\right) \times \left(10^{12}\ \text{ly}^3\right) \\ &= -4\ \pi\left(11.2\ x\ 10^{-21}\ \text{ly}^3/s^2\right) \\ &= -1.41\ x\ 10^{-19}\ \text{ly}^3/s^2\end{aligned}$$

A light year is 460 730 472 580.8 km (Wikipedia). This is approximately $46\ \text{x}10^{10}$ km. Then

$$1\ \text{ly}^3 = (46\ 10^{10})^3\ \text{km}^3 = 46^3 \times 10^{30}\ \text{km}^3 = \text{approx}\ 10^5 \times 10^{30} = 10^{35}\ \text{km}^3.$$

Then

$$\delta\ddot{\text{V}} = -\frac{1.40605\times10^{-19} \times 10^{35}\ \text{km}^3}{s^2} = -\frac{1.40605\times10^{16}\ \text{km}^3}{s^2}.$$

Then if I have not missed an important assumption or gotten a conversion wrong, the initial **acceleration** of shrinkage in a **static** Galaxy would be $-1.4 \times 10^{16}\ \text{km}^3/s^2$. The shrinkage of a tiny fraction of a light year cubed turns out to be a very large number of kilometers cubed.

As shrinking proceeds, it must be necessary to account for increasing density, which would be linear with shrinking volume, so perhaps acceleration of volume would also increase linearly with shrinking volume.