Needham, VDGAF, Chapter 3, Exercise 9, p. 25

Palmer, January 17, 2022, 9:50 am PST

Out[271]=

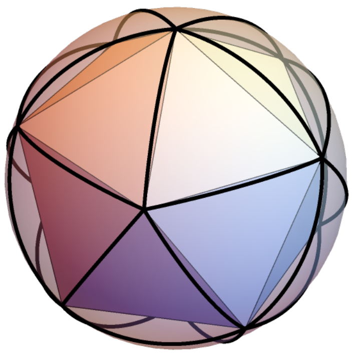

Figure 1. Icosahedron with triangular faces projected to sphere

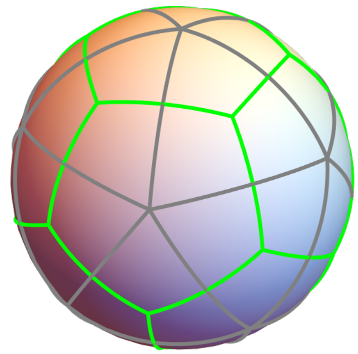

Figure 2. Dodecahedron projected to sphere (green) and lines from centres to midpoints of edges (black)

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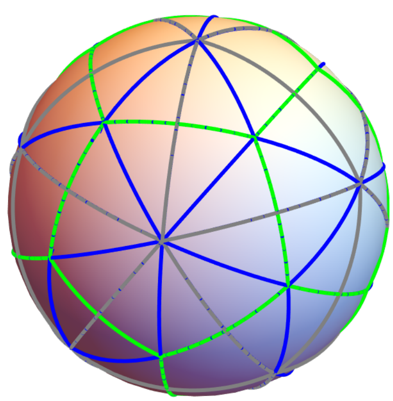

Figure 3. Tesselation for one spherical pentagon (green border) and three triangles (gray borders) as in [3.1]

Regarding the caption under [3.1], the instructions for creating the illustrated tesselation read *by joining its centre to the midpoints of its edges.* The phrase is here rewritten as *by joining every vertex of every triangle to the midpoint of its opposite edge.* The same method is applied to the dodecahedron in part (ii).

(i) Explain why the shadows of the icosahedral edges are arcs of great circles on the sphere, thereby

creating genuine spherical triangles.

Without looking at the icosahedron, one can deduce that a shadow can only be an arc of a great circle if it lies in the same plane as an edge, and it must be a plane that passes through the origin. Hence, rays in the plane that originate from the origin and are (metaphorically) blocked by the edge project as shadow arcs of great circles onto the sphere. See Figure 1.

(ii) Verify that if we instead inscribe a dodecahedron, centrally project its pentagons onto the sphere, then join their [vertices] *to the midpoints of the* [opposite] *edges, we obtain the same tesselation* [as illustrated in [3.1]].

To plot Figure 3, joins between vertex and midpoints of edges were accomplished as follows: First, every edge of the dodecahedron was extended into a complete great circle. Then at each vertex, the great circles were rotated by $\pi/3$ or $-\pi/3$ about the vertex vector anchored at the origin. The method was applied to only one pentagon, but the effect was to create full tesselation in the spherical pentagon and three overlapping spherical triangles of the icosahedron. There are 10 congruent tiles in the pentagon and six in each of the three triangles. The apparently discontinuous blue segments in Figure 3 actually were plotted as complete great circles. They were given a slightly smaller radius in order to reveal the green and gray borders of the pentagons and triangles.

(iii) Into how many congruent triangles has the sphere (of area $4\pi R^2$) been divided? So what is the area $\mathcal{A}$ of each triangle?

Since the icosahedron projected to the sphere has 20 triangles, each divided into 6 congruent triangles, the total number is 120. Then the area of each is $4\pi R^2/120 = \pi R^2/30$. If the sphere happens to be the unit sphere, the area of each is $\pi/30$, which equals the angular excess.

(iv) By observing how many like angles come together at each vertex, deduce that the angles are $(\pi/2)$, $(\pi/3)$, and $(\pi/5)$. Hence, calculate the angular excess ε of each triangle.

Like angles come together at vertices in bunches of 4, 6, and 10. The angles of each small triangle are thus $2\pi/\{4, 6, 10\} = \{\pi/2, \pi/3, \pi/5\}$. From the equation at the top of p. 8, the angular excess is $\varepsilon = \pi/2 + \pi/3 + \pi/5 - \pi = 31\pi/30 - \pi = \pi/30$. This is in agreement with part (iii).

(v) Verify that the previous two answers are in accord with Harriot's theorem, (1.3).

By Harriot's theorem, $\varepsilon = \frac{1}{R^2}\mathcal{A}$. Then by part (iii) $\varepsilon = \frac{\pi R^2/30}{R^2} = \pi/30$. In part (iv), the answer was calculated with Harriot's theorem, so it must be in accord with the theorem. Since both give the same value for ε, both answers are in accord with Harriot's theorem.