Needham, VDGAF, Chapter 3, Exercise 8, p. 24
Palmer, December 8, 2021, 5:40 pm PST

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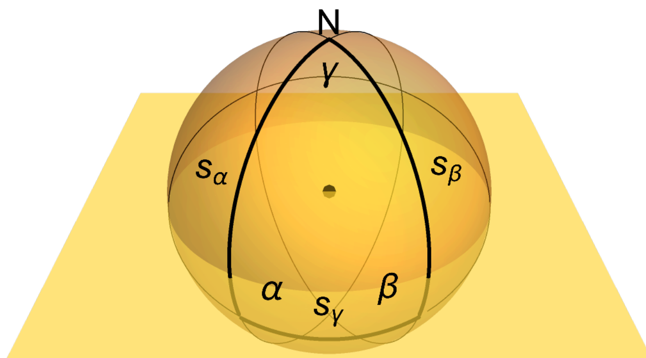

`Figure 1. Triangle of sides each < π contained within hemisphere`

8 If the sides of a triangle on the unit sphere $\mathbb{S}^2$ are each less than π, show that the triangle is contained within a hemisphere. [Hint: while it is mathematically immaterial, it can be psychologically helpful to imagine that the sphere has been rotated so that one vertex is at the north pole.]

From the vertex at N, plot a side s_α as a meridian in any direction, but less than π in length. If the rotation were to exceed π, the side would cross the south pole and pass into a new hemisphere. Plot a second meridian-side s_β of the same length. Let the angle between them approach, but not reach π. Now plot a side s_γ connecting the endpoints of s_α and s_β with a segment of a **great circle**. If the endpoints lie near the equator, they are almost π apart measured on the unit sphere. If they are near the south pole, they are very close together. As the angle γ approaches π and sides S_α, and S_β approach π, the triangle approaches the area of the hemisphere. At the same time, angles α and β approach π, because the arc between the endpoints is a segment of a great circle and the segment s_γ is approaching congruence with the great circles of the other two sides, but α and β do not reach π. α, β, and γ are all less than π. From p. 9, $\mathcal{A}(\triangle) = \mathcal{A}(\triangle) = R^2(\alpha + \beta + \gamma - \pi)$. We can now substitute our values:

$$\mathcal{A}(\triangle) = R^2(\alpha + \beta + \gamma - \pi) < \left(R^2(\pi + \pi + \pi - \pi) = 2\,\pi R^2\right)$$

Hence, the area of a triangle with sides each less than π is less than $2\,\pi R^2$, which is the area of a hemisphere. The triangle is contained within the hemisphere.

Vasco used a more direct proof. Once the ends of the meridian-sides are joined by a great circle segment to make the third side of the triangle, one has a choice between two triangles, each with a base in the great circle containing the third side. One base is longer than the other so long as the meridian sides do not lie in a single plane (angle π). To remain within a single hemisphere, just choose the triangle with the shorter base. The shorter base must be less than π in length because it is the segment of some great circle. If the two bases were equal, they would both be of length π.

To state this more succinctly, any two vertical arcs of great circles arranged about the z-axis axis at an angle not equal to π are guaranteed to lie in the same hemisphere, so long as the hemisphere is not predetermined. Then if the two arcs have length less than π, it is also inevitable that the shortest arc of a great circle connecting their end points lies in the same hemisphere and has length less than π. Hence, the triangle formed of the three sides lies within a hemisphere.

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