Needham, VDGAF, Chapter 3, Exercise 7, p. 24
Palmer, December 5, 2021, 1:10 pm PST

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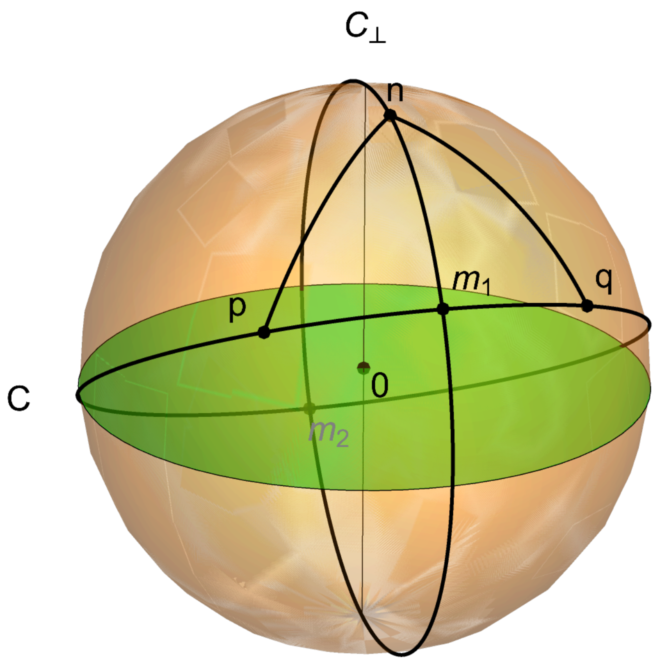


Figure 1. Great circle C, bisector $C_\perp$

*Let p and q be distinct, nonantipodal points on a sphere,and consider the unique great circle C through them. Let $m_1$ and $m_2$ be the midpoints of the two arcs into which C is split by p and q. Show that the locus of points that are equidistant from p and q is the great circle through $m_1$ and $m_2$ that cuts C orthogonally: this is the generalized "perpendicular bisector" of pq. (Hint: while it is mathematically immaterial, it can by psychologically helpful to imagine that the sphere has been rotated so that p and q lie on the equator.)*

By "equidistant" the author must mean equidistant on the sphere, i.e. distance measured as the length of the arcs (p, n) and (q, n) on the sphere rather than the Euclidean distance |n-p| or |n-q|. We are given that D(p, n) = $\theta$ r = D(q, n), where $D(v_1, v_2)$ is the distance on the sphere between the vectors $v_1$ and $v_2$, r is the radius of the sphere, and $\theta$ is the angle between the vectors p and n. Since it is given that the D($p$, $m_1$) = $D(q, m_1)$, $\theta$ must also be the angle between q and n. It is also given that D($p$, $m_2$) = $D(q, m_2)$.

The triplet (p, n, q) defines a triangle on the sphere. We will not be concerned with other triangles on the sphere. Stipulate that as n traverses the sphere from $m_1$ to $m_2$, side D(p, n) remains equal to side D(q, n) and the intersection of the two arcs at n remains on the sphere. Shrink the triangle by letting p, q, and n $\longrightarrow M_1$. As the triangle shrinks, arc $(p, n) \asymp n - p$ and arc $(q, n) \asymp n - q$. Spherical triangle T = (p, n, q) approaches an Euclidean isosceles triangle, while the path of n over arc (n, $m_1$) ultimately becomes congruent with the perpendicular bisector. This shows that the path of n, which is the locus of points equidistant from p and q, is the great circle through $m_1$ and $m_2$ that cuts C orthogonally.