Needham, VDGAF, Chapter 3, Exercise 5, p. 24
Palmer, June 6, 2026, 10 pm PST

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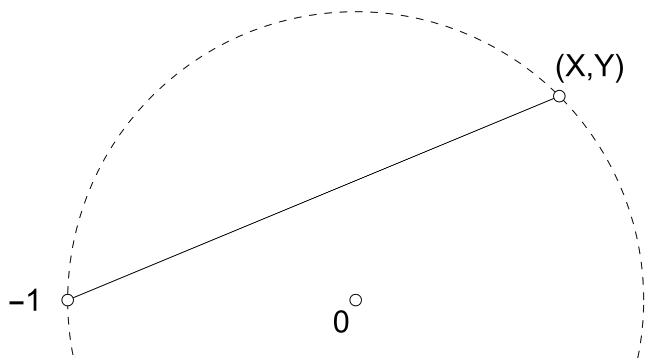


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Figure 1. Construction for generating rational coordinates
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(i) Let L be the line y = m(x+1) through the point (-1, 0) of the unit circle C, and let (X, Y) be the second intersection point of L and C. Prove that

$$X = \frac{1-m^2}{1+m^2} \quad and \quad Y = \frac{2m}{1+m^2}.$$

$$m = \frac{Y}{X+1} = \frac{\frac{2m}{1+m^2}}{\frac{1-m^2}{1+m^2}+\frac{1+m^2}{1+m^2}} = \frac{2m}{2} = m$$

This is what we wanted.

(ii) Deduce that if the slope m = (q/p) is rational, then so are X and Y.

Numbers q and p must be integers by the definition of rational fraction. Substitute (q/p) into the equation for X from part (i).

$$X = \frac{1-(q/p)^2}{1+(q/p)^2} = \frac{p^2-q^2}{p^2+q^2}$$

$(p^2 - q^2)$ and $(p^2 + q^2)$ are integers because p^2 and q^2 are integers. Therefore the quotient X is rational.

Substituting (q/p) for m, one obtains $Y = \frac{2\,\mathrm{pq}}{p^2+q^2}$. Both the product in the numerator and the sum in the denominator are integers, so the quotient Y is rational.

(iii) Deduce that if (a,b,h) is an arbitrary Pythagorean triple, then

$$\frac{a}{h} = \frac{p^2-q^2}{p^2+q^2} \qquad \text{and} \qquad \frac{b}{h} = \frac{2\,\text{pq}}{p^2+q^2}$$

If the equations are true, then $\left(\frac{p^2-q^2}{p^2+q^2}\right)^2 + \left(\frac{2\,\text{pq}}{p^2+q^2}\right)^2 = 1$ because cos ^ 2 + $\sin^2 = 1$.

$$\left(\frac{p^2-q^2}{p^2+q^2}\right)^2 + \left(\frac{2\,\text{pq}}{p^2+q^2}\right)^2 = \left(\frac{p^4-2\,p^2\,q^2+q^4+4\,p^2\,q^2}{p^4+2\,p^2\,q^2+q^4}\right) = \left(\frac{p^4+2\,p^2\,q^2+q^4}{p^4+2\,p^2\,q^2+q^4}\right) = 1$$

This is what we wanted. The equations are true.

(iv) Deduce that the most general Pythagorean triple is given by the following formula stated by Euclid:

$$a = \left(p^2 - q^2\right) r \quad \text{and} \quad b = 2\,pqr \quad \text{and} \quad h = \left(p^2 + q^2\right) r,$$

where p, q, and r are arbitrary integers.

By calculating the Pythagorean squares for $a^2 + b^2 = h^2$, one can see that the r^2 factors cancel out of the equation. Then one only has to show that $\left(p^2 - q^2\right)^2 + (2\,\text{pq})^2 = \left(p^2 + q^2\right)^2$. Then

$$\left(p^4 - 2\,p^2\,q^2 + q^4 + 4\,p^2\,q^2\right) + \left(4\,p^2\,q^2\right) = p^4 + 2\,p^2\,q^2 + q^4,$$

which is almost self-evident. Since p and q can be arbitrary integers, the equation is true for any p and q. Let $\tilde{a} = \left(p^2 - q^2\right)$, $\tilde{b} = 2\,\text{pq}$, and $h = \left(p^2 + q^2\right)$. Then given the now proven triplet equation $\tilde{a}^2 + \tilde{b}^2 = \tilde{c}^2$, the equation is still true if one multiplies through by an arbitrary integer r to obtain $\tilde{a}^2\,r + \tilde{b}^2\,r = \tilde{c}^2\,r$. Given that p, q, and r are all arbitrary integers, this must be the most general Pythagorean triple.