Needham, VDGAF, Chapter 3, Exercise 3, p. 24

Palmer, December 1, 2021, 9:15 am PST

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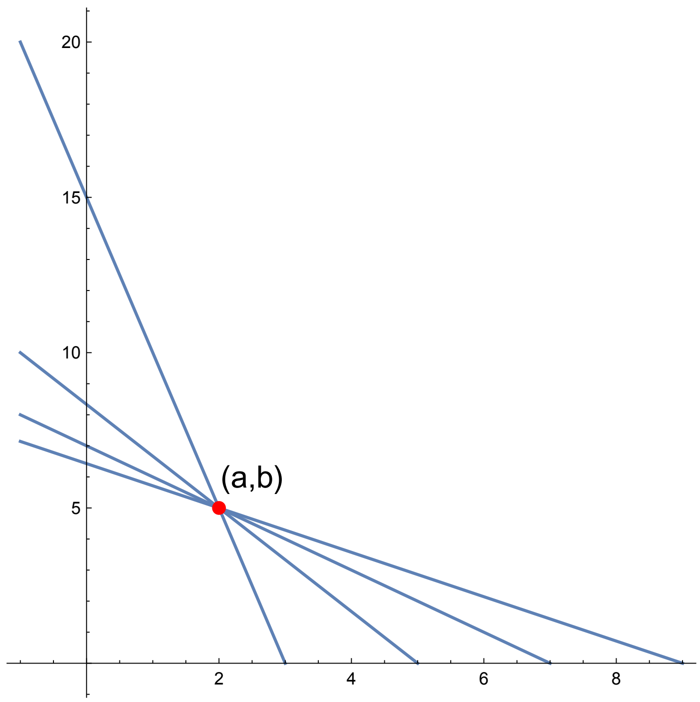

Figure 1. Arbitrary lines through (a,b) in Quarter I

3. Let L be a general line through the point (a,b) in the first quadrant of $\mathbb{R}^2$*, and let* $\mathcal{A}$ *be the area of a triangle bounded by the x-axis, the y-axis, and L.*

(i) Use ordinary calculus to find the position of L that minimizes $\mathcal{A}$ *and show that* $\mathcal{A}_{min} = 2\,ab$.

Figure 1 shows some of the lines that might be plotted through the point (a, b) in red.

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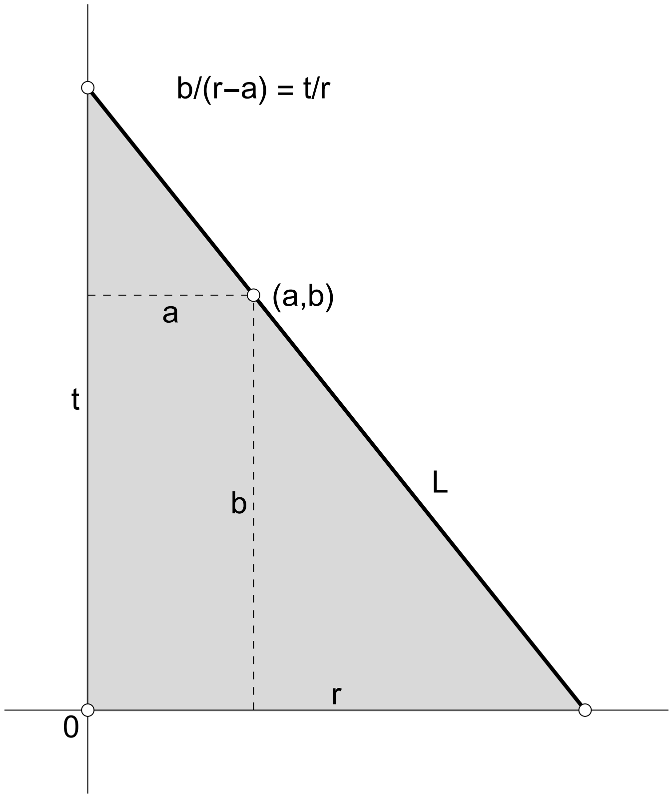

Figure 2. Area under L in Quarter I

Since the point of intersection is fixed at (a, b), alternative lines can only be created by varying the intercepts, which may be written (r, 0) and (0,t) for the x and y intercepts. Where r > a, t > b, the equation for the area is $\mathcal{A} = (1/2)\, r\, t$. From Figure 2, above, we have

$$\frac{b}{r-a} = \frac{t}{r} \qquad r = \frac{-at}{b-t}$$

Substitute r into the area equation.

$$\mathcal{A} = -(1/2)\, \mathrm{at}^2 \big/ (b - t)$$

To obtain $\mathcal{A}_{\min}$, find $\frac{d\mathcal{A}}{dt}$ and set it equal to zero.

$$\frac{d\mathcal{A}}{dt} = \left(\mathrm{at}^2 - 2\,\mathrm{abt}\right) \big/ 2\,(b - t)^2 = 0 \Longrightarrow t^2 - 2\,\mathrm{bt} = 0$$

$$t(t\text{-}2b) = 0 \quad \Longrightarrow \quad t = 0, t = 2b$$

The y intercept does not go to zero in this exercise, so when $\mathcal{A}$ is a minimum, the y-intercept is 2b. One could plot a line through (0, 2b) and (a, b) to obtain the x-intercept, or one can solve for r. Substitute for t into the equation in Figure 2 to find r in terms of a and b, and substitute into $\mathcal{A} = (1/2)\, rt$.

$$r = -at/(b-t) = -a\, 2b/(b-2b) = -2ab/-b = 2a$$

$$\mathcal{A}_{min} = (1/2) \times (2\, a) \times (2\, b) = 2\, ab$$

Figure 3 compares the arbitrary line L from Figure 2 with the line of minimum area for (a, b) = (2, 5).

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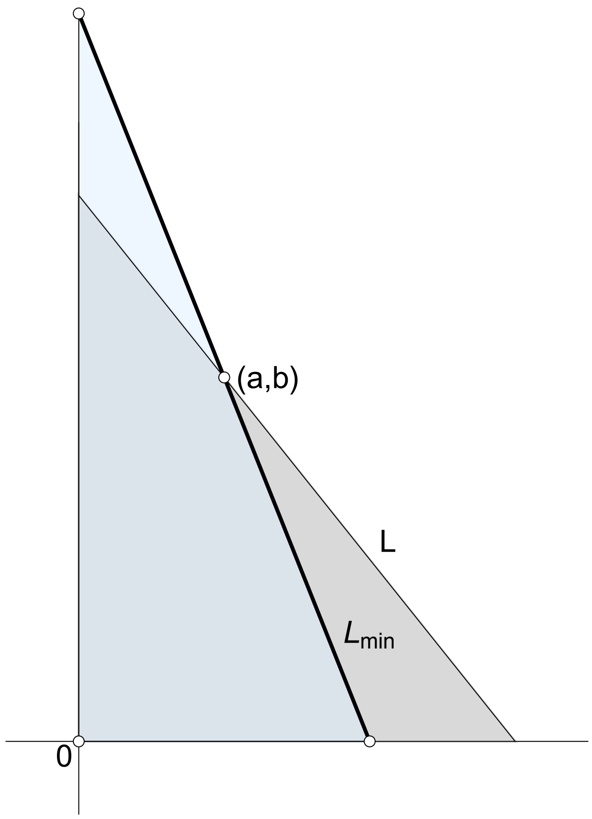

Figure 3. Min Area (Blue) under L in Quarter I

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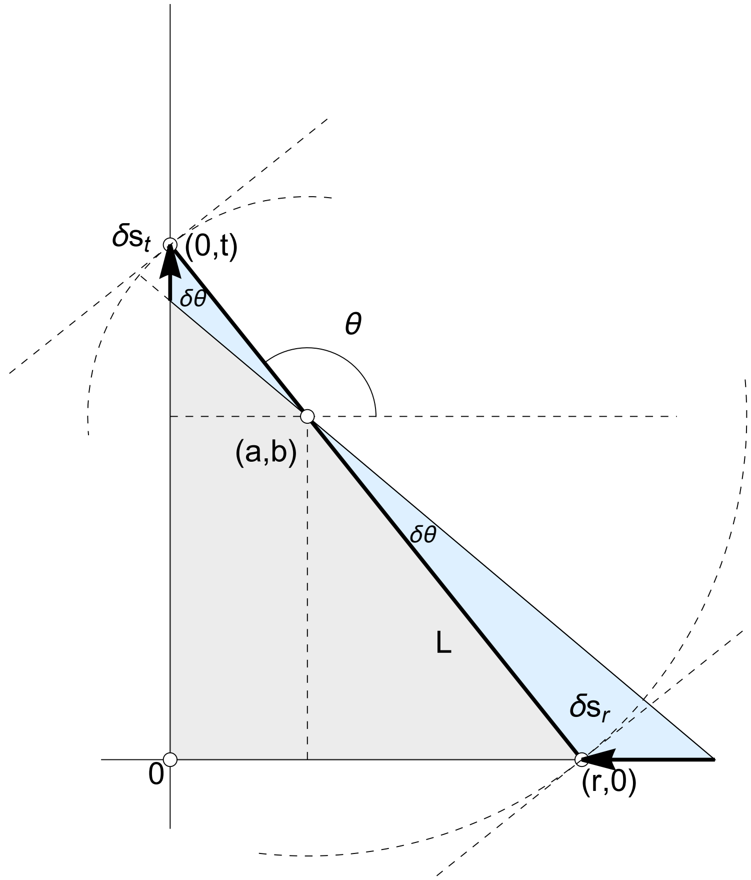

Figure 4. $\delta\mathcal{A}$ sum of two triangles in light-blue ultimate sectors

(ii) Use Newtonian reasoning to solve the problem instantly, without calculation! (Hints: Let $\delta\mathcal{A}$ be the change in area resulting from a small (ultimately vanishing) rotation $\delta\theta$ of L. By drawing $\delta\mathcal{A}$ in the form of two triangles, and observing that each triangle is ultimately equal to a sector of a circle, write down the ultimate equality for $\delta\mathcal{A}$ in terms of $\delta\theta$. Now set $\delta\mathcal{A} = 0$.

Let $\delta\mathcal{A}_t$ be the area of the small blue triangle which has a vertex at (a,b) and one side equal to δt (an infinitesimal change on the y-axis, shown with a small bold arrow in Figure 4). The area of this triangle will be subtracted because it reduces $\mathcal{A}$. Let $\delta\mathcal{A}_r$ be the area of the small triangle which has one side equal to δr (an infinitesimal change on the x-axis).

$$\delta\mathcal{A} = -\delta\mathcal{A}_t + \delta\mathcal{A}_r \qquad (1)$$

Since $\delta\mathcal{A}_t$ and $\delta\mathcal{A}_r$ are ultimately equal to sectors of the circle |z| = |(0,t) - (a, b)| = |(r,0) - (a,b)|, one can let $\delta\theta$ approach 0 and write $\delta t \asymp 0$, $\delta s_t \asymp \delta\theta \cdot R_t$, and $R_t = |\,(0, t) - (a, b)\,|$, and similarly with δs_r and R_r. δs_i ultimately equals a side (base or height, depending on one's perspective) tangent to the circle of one of the sector triangles.

$$\delta s_t \doteq \delta\theta\ |(0, t) - (a, b)| = |-a,\ t - b|$$

$$\delta s_r \doteq \delta\theta\ |(r, 0) - (a, b)| = |r - a, -b|$$

Then $\delta\mathcal{A}_t \doteq (1/2)\,\delta s_t\, R_t \doteq (1/2)\,(\delta\theta \cdot R_t)\, R_t\ =\ \delta\theta\, R_t^2$, so

$$\delta\mathcal{A}_t \doteq (1/2)\,\delta\theta\ |-a,\ t - b|^2 = (1/2)\,\delta\theta\left(a^2 + b^2 - 2\,\mathrm{bt} + t^2\right)$$

Similarly,

$$\delta\mathcal{A}_r \doteq (1/2)\,\delta\theta\ |r - a, -b|^2 = (1/2)\,\delta\theta\left(a^2 + b^2 - 2\,\mathrm{ar} + r^2\right)$$

Now to find t and r when L is in the minimum position, set $\delta\mathcal{A} = 0$, which means that $\delta\mathcal{A}_t = \delta\mathcal{A}_r$, and cancel (1/2)$\delta\theta$ on both sides. Letting $\delta\mathcal{A} = 0$ is analogous to letting $\frac{d\mathbb{A}}{d\theta} = 0$ in calculus.

$$\left(a^2 + b^2 - 2\,\mathrm{bt} + t^2\right) = \left(a^2 + b^2 - 2\,\mathrm{ar} + r^2\right)$$

$$-2\,\mathrm{bt} + t^2 = -2\,\mathrm{ar} + r^2$$

$$t(t - 2\,b) = r(r - 2\,a)$$

These are equal when $t = 2\,b$ and $r = 2\,a$, which is what we obtained using calculus. Then, $\mathcal{A}_{\min} = (1/2) \times (2\,b) \times (2\,a)\ =\ 2\,\mathrm{ab}$, which is what we wanted.

It is obvious in Figure 5, that $\delta\mathcal{A}_r$ is greater than $\delta\mathcal{A}_t$, so the positive anticlockwise rotation of L away from the position of the minimum area increases the area by the difference. If the direction of rotation away from the minimum were reversed to clockwise, then $\delta\mathcal{A}_t > \delta\mathcal{A}_r$, but their signs are reversed, so the effect is still to increase the area.

Endnote

The following "answer," which occurred to me while working on part (i), is not intended as an answer to either (i) or (ii). It is neither calculus nor Newtonian. I would characterize it as a geometric answer in which the solution emerges empirically when, following rotation of L about (a,b) a rotation of triangle T about (a,b) creates a new rectangle based on the horizontal line through a. The triangle of minimal area has the midpoint of the hypotenuse at (a,b) and obvious area 2ab.

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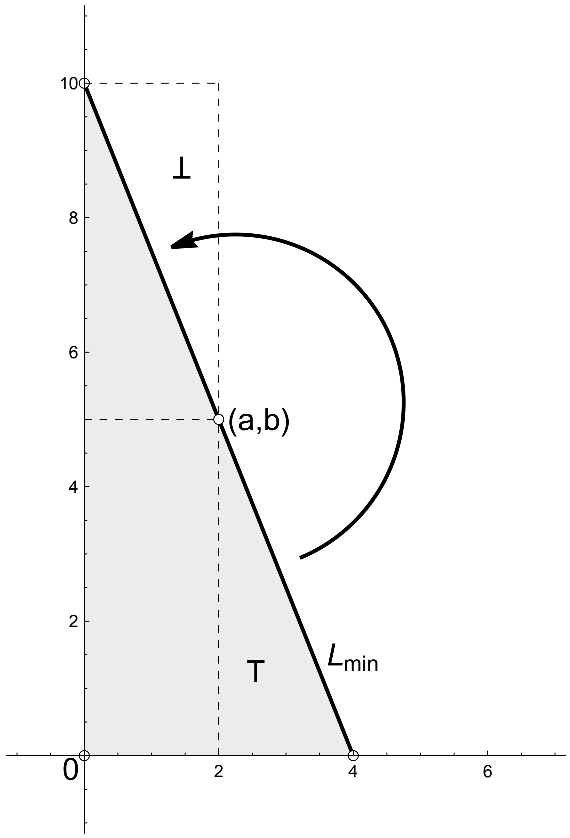

Endnote figure. Min area L in Quarter I

Figure 4 shows instantly that $\mathcal{A}_{min}$ = 2 ab without the use of Newtonian reasoning. Rotating L until the y-intercept equals 2b or the x intercept equals 2a results in $\mathcal{A}_{min}$. That $\mathcal{A}_{min}$ = 2 ab is shown “instantly” by rotating triangle T by π radians about (a, b).