Needham, VDGAF, Chapter 3, Exercise 14, pp. 27-28
Palmer, January 4, 2022, 10 am PST

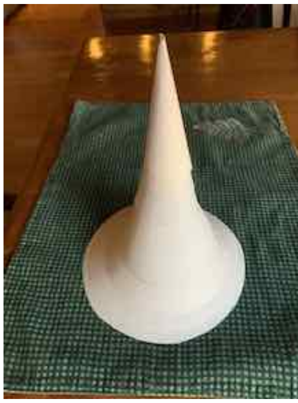 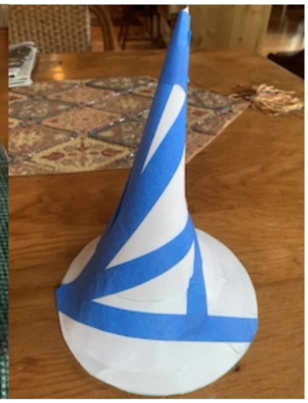

Photo 1 Photo 2

(i) By following the instructions that accompany [5.3], page 53, construct your own personal pseudo-sphere out of discs of radius R---the more cones, the better; the bigger, the better!

See Photo 1.

(ii) With the sole exception of the meridian geodesic that heads straight up the surface---tracing a tractrix generator of the surface of revolution---note that every geodesic initially heads up the pseudosphere but then turns around and comes back down the pseudosphere, ultimately returning to the base circle.

See Photo 1.

(iii) Construct a right-angled geodesic triangle $\triangle$, measure its angles, and hence estimate its angular excess $\varepsilon(\triangle)$. Estimate as best you can its areas, $\mathcal{A}(\triangle)$. Hence estimate the (constant) curvature $\mathcal{K}$ of your pseudosphere, using

$$\mathcal{K} = \frac{\varepsilon(\triangle)}{\mathcal{A}(\triangle)}$$

For the triangle with a right angle, see photo 2. The angles measured with a protractor are 90, 51 and 27 degrees (total 168), which translates to $\pi/2$, $17\pi/60$, and $3\pi/20$ radians (total $14\pi/15$). The angular excess is $-\pi/15$. The base length, measured with a straight rule against the flattened (by squashing) pseudosphere, is 5.5 cm and the height is 7.2 cm, for an estimated area of 19.8 cm^2. Hence, the estimated constant curvature (rounded to 3 decimal places) is $\mathcal{K} = -0.011$.

Table 1. Calculations in Mathematica

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angles: {90, 51, 27}
angle sum: 168
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radians: $\left\{\frac{\pi}{2}, \frac{17\pi}{60}, \frac{3\pi}{20}\right\}$

angle sum radians: $\frac{14\pi}{15}$

$\varepsilon(\triangle)$: $-\frac{\pi}{15}$

$\varepsilon(\triangle)$ as percent of π: $-\frac{20}{3}\%$

$\mathcal{A}_E$: 19.8 cm^2

$\mathcal{K}=\varepsilon(\triangle)/\mathcal{A}_E$: -0.0105778

(iv) The larger the triangle, the larger (i.e. more negative) the value of ε(△), making its measurement easier and more accurate. But the tradeoff is that it becomes harder to accurately estimate the area $\mathcal{A}$(△). To overcome this difficulty do the following. Make narrow strips of the same kind that you use to generate geodesics, but create them all with the same (accurately measured) width, W, perhaps 1/4 inch. Now fill your △ with these strips, cutting them off when they hit an edge. Remove the strips and lay them end-to-end on a flat surface, and measure the total length, L. Then $\mathcal{A} \asymp LW$.

The triangle in Photo 2 was filled with strips of transparent green garden tape of width 1.2 cm. The transparency of the tape enabled measurements from side to side without cutting the strips. Five strips were used. Due to some clumsiness of the experimenter, the first and last segments used only partial widths. The measurements of length of the five segments arranged from bottom to top in as follows:

{7.3, 5.1, 3.5, 2.2, .4}

Multiplying 1.2 cm times the total length gives 19.16 cm^2, which compares well to the 19.8 cm^2 calculated from base and height of the triangle. Corrections of –(.4·7.3) and –(.3·.4) were added for overlaps of the first and last strips outside the triangle. The curvature calculated using the two calculations of area agree to three decimal places at $\mathcal{K}$ = –0.011.

Table 2. Calculation of area with $\mathcal{A}$ = LW

area calculated with strips: 19.16

$\mathcal{K}$ calculated with strips: -0.0109311

(v) Repeat (iii) with several more triangles, but no longer restrict them to be right-angled, because (iv) now

makes it easy to measure $\mathcal{A}(\triangle)$ for any shape of triangle. Verify that (within experimental error) all triangles yield the same value of $\mathcal{K}$.

I measure one more triangle (not shown) rather than several. My model is not accurate enough to support the whole exercise. The second triangle has one strip of width .9 and four strips of width 1.2. The angles are {23, 23, and 128} degrees. $\mathcal{A} = 3.71\ \text{cm}^2$ and $\mathcal{K} = -0.028$. The lengths of the strips are

lengths = {0.5, 1.0, 1.2, 0.8, 0.3}

A small correction of $-(0.3\cdot 0.3) = -0.09$ is added for overlap on the last (top) strip.

Table 3. Calculations of $\mathcal{A}$ and $\mathcal{K}$ for second triangle

Lengths: {0.5, 1., 1.2, 0.8, 0.3}

$\mathcal{A}$2: 3.71 cm^2

angles T2, degrees: {23, 23, 128}

angles T2, radians: $\left\{\frac{23\pi}{180}, \frac{23\pi}{180}, \frac{32\pi}{45}\right\}$

angle sum, radians: $\frac{29\pi}{30}$

ε2($\triangle$): $-\frac{\pi}{30}$

$\mathcal{K}$2: -0.0282263

(vi) Assuming that

$$\mathcal{K} = -\frac{1}{R^2},$$

estimate R, and compare this to the actual radius of the discs you used to construct your pseudosphere.

The actual radius of the discs is 9 cm. From triangle 1, $\mathcal{K} = -.011$ by both methods of measuring area.

Triangle 1

$$R^2 = -\frac{1}{\mathcal{K}} \implies R = \frac{i}{\sqrt{\mathcal{K}}} = \frac{i}{\sqrt{-0.011}} = \frac{1}{\sqrt{0.011}} = 9.53\ \text{cm}$$

Triangle 2

$$R^2 = -\frac{1}{\mathcal{K}} \implies R = \frac{i}{\sqrt{-0.028}} = \frac{1}{0.167} = 5.98\ \text{cm}$$

The calculation of R for triangle 1 is close enough. The calculation for triangle 2 is off by 37 percent, which is easily explained by irregularities in the model pseudosphere causing errors in measuring angles and lengths of strips.