Needham, VDGAF, Chapter 3, Exercise 10, p. 26
Palmer, December 11, 2021, 12 midnight PST

Out[469]=

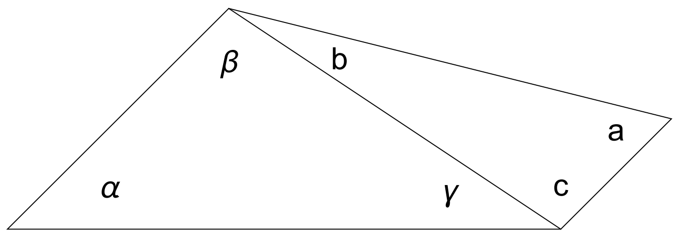

Figure 1. Quadrilateral with diagonal

10 (i) Prove that in Euclidean Geometry the angle sum of a quadrilateral is 2π.

Plot a diagonal that divides the quadrilateral into two triangles (Figure 1). We know that

$$(\alpha + \beta + \gamma) = \pi = (a + b + c),$$

so

$$(\alpha + \beta + \gamma + a + b + c) = 2\,\pi$$

Add the angle sums at the four vertices of the quadrilateral.

$$(\alpha + (\beta + b) + a + (c + \gamma) = (\alpha + \beta + \gamma + a + b + c) = 2\,\pi$$

QED

(ii) If Q is a geodesic quadrilateral on the sphere of radius R, its angular excess is therefore

$$\varepsilon\,(Q) = (\textit{angle sum of } Q) - 2\,\pi$$

By drawing a diagonal of Q, thereby splitting it into two geodesic triangles, deduce that (1.3) generalizes to

$$\varepsilon\,(Q) = \frac{1}{R^2}\,\mathcal{A}\,(Q).$$

From p. 8, $\varepsilon = \frac{1}{R^2}\,\mathcal{A}$. Let the geodesic triangles be $\triangle_1$ and $\triangle_2$. Then,

$$\mathcal{A}(Q) = \mathcal{A}(\triangle_1) + \mathcal{A}(\triangle_2)$$

$$\varepsilon(\triangle_1) = \frac{1}{R^2}\,\mathcal{A}(\triangle_1), \qquad \varepsilon(\triangle_2) = \frac{1}{R^2}\,\mathcal{A}(\triangle_2)$$

$$\varepsilon(Q) = \frac{1}{R^2}\,\mathcal{A}(\triangle_1) + \frac{1}{R^2}\,\mathcal{A}(\triangle_2) = \frac{1}{R^2}\left[\mathcal{A}(\triangle_1) + \mathcal{A}(\triangle_2)\right] = \frac{1}{R^2}\,\mathcal{A}(Q)$$

(iii) Prove (2.2), and hence generalize (ii) to geodesic n-gons on the sphere.

$$\varepsilon(n - \text{gon}) = (\text{angle sum}) - (n-2)\,\pi \qquad (2.2)$$

If one arranges a series of Euclidean n-gons, they can all be divided into a minimal number (n-2) of triangles by plotting triangle base-lines for every three vertices, and for larger n-gons, repeating the process with the n-gons that arise inside the originals. Then (2.2) follows from two facts:

1. The sum of the interior angles of an Euclidean n-gon equals the angle sum (n-2) π of the (n-2) interior triangles, as seen, for example, in the quadrilateral of part (i).
2. The angular excess of a geodesic triangle equals the sum of interior angles minus π.

Hence, the angular excess ε(n-gon) equals the angle sum of the geodesic n-gon minus the sum (n-2)π of angles of an Euclidean n-gon.