Needham, VDGAF, Chapter 3, Exercise 1, p. 24.
Palmer, November 16, 2021, 11:40 pm PST

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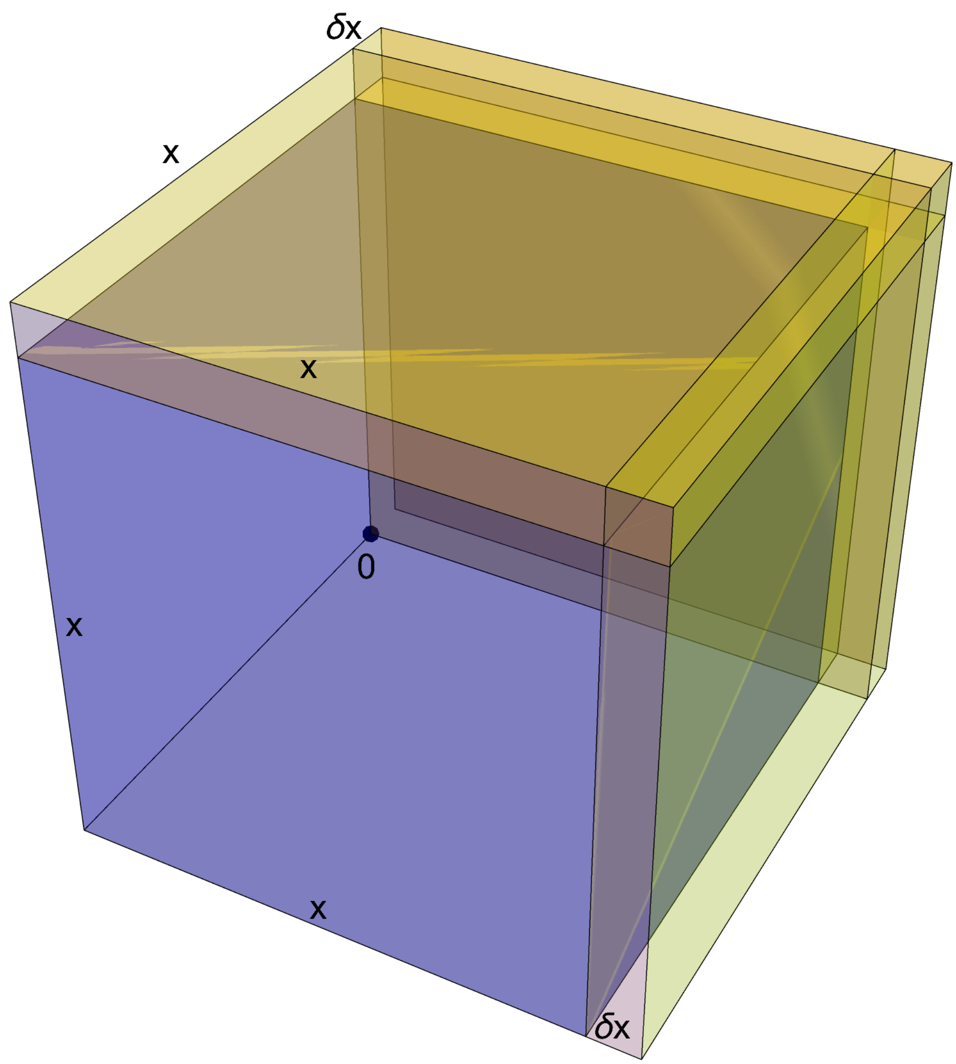

Figure 1. Cube of side x inside cube of side x + δx

1. *(This is a model of how an "ultimate equality" ... becomes an equality.) Sketch a cube of side x, hence of volume $V = x^3$. In the same picture, keeping one vertex fixed, sketch a slightly larger cube with side $x + \delta x$. If δx is the resulting increase in volume, use your sketch to deduce that as dx vanishes,*

$$\delta V \asymp 3 x^2 \,\delta x \Longrightarrow \frac{dV}{dx} \asymp \frac{\delta V}{\delta x} \asymp 3 x^2 \Longrightarrow \frac{dV}{dx} \asymp 3 x^2$$

But the quantities in the final ultimate equality are independent of δx, so they are equal:

$$\left(x^3\right)' = \frac{dV}{dx} = 3\,x^2.$$

Figure 1 shows the original cube in blue anchored at the origin. The sides of the cube are expanded by δx as shown by the yellow extensions, which turn purple and green where they overlay the blue. The volume δV of the yellow extensions is a three sided corner with walls of thickness *δx.*

Enlarge all sides of the cube by δx while leaving the original cube of side x inside the new cube of side (x + δx) with one corner fixed at the origin. Then remove the original cube, leaving an extruded corner shape with three sides of length and width (x + δx) and thickness δx. The volume δV of this shape is $(x + \delta x)^3 - x^3$. Let δx go to an infinitesimal thickness. The resulting δV is three square sides of area x^2 and thickness δx described below in the first term of (1) plus three beams of thickness δx by δx along the vertices and one corner cube with sides of δx. The second and third terms of (1) describe these intersections of the sides. These intersections are the three beams and the $(\delta x)^3$ corner at their intersection at the upper right of Figure 1. The beams and the $(\delta x)^3$ corner shrink to three lines and a point and vanish in the second and third powers of δx.

$$V + \delta V = (x + \delta x)^3 = x^3 + 3\,x^2\,\delta x + 3\,x(\delta x)^2 + (\delta x)^3$$
$$\Longrightarrow$$
$$\delta V = \left(V + \delta V - x^3\right) = \mathbf{3\,x^2\,\delta x} + 3\,x(\delta x)^2 + (\delta x)^3 \qquad (1)$$

Retain only the first term of (1) and divide by δx.

$$\frac{\delta V}{\delta x} \asymp 3\,x^2$$

$$\Longrightarrow \frac{dV}{dx} = 3\,x^2 = \left(x^3\right)'$$

As Needham stated, the quantities in the ultimate equality are independent of δx and equal, so $\frac{dV}{dx} = 3\,x^2$.