Notes on Needham, VDGF, Chapter 29.
G. Palmer, July 26, 2025, 6 pm PDT

Out[∘]=

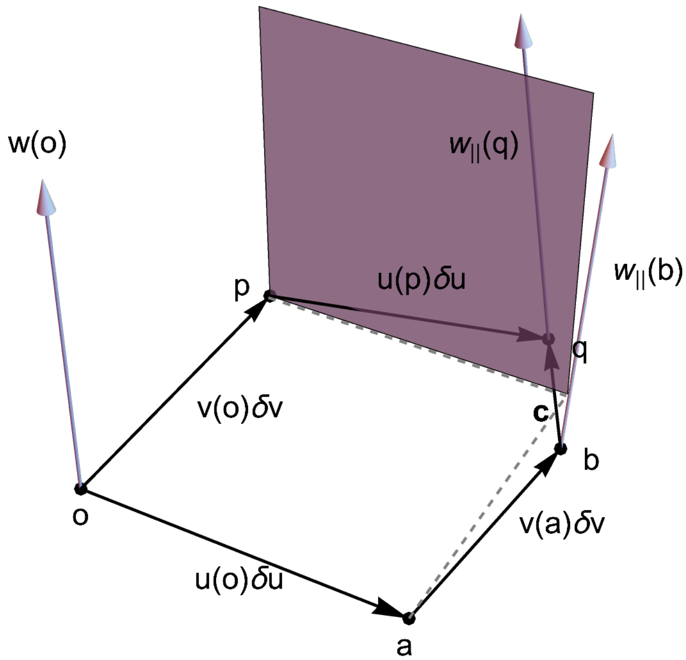

Figure 1. Holonomy on commutator of parallel-transport on general surface. See [29.6], [29.7], and [29.8] in Needham, *VDGF*.

In Figure 1, the tubular arrows represent parallel-transport from the origin to points b and q. The arrows labeled $\boldsymbol{w}_{||}(b)$ and $\boldsymbol{w}_{||}(q)$ give some indication of the holonomy on **c**, but the actual holonomy revealed by the traversal of $\boldsymbol{w}_o$ on **c** can only be calculated by placing the vectors $\boldsymbol{w}_{||}(b)$ and $\boldsymbol{w}_{||}(q)$ at the origin or other common anchor point and finding the difference in their directions. Curvature on the surface has given vector fields **u**(p)δu and **v**(a)δv small rotations ($\pi/32$) in two directions, resulting in both lying outside the boundaries of the rectangle floor completed by the two dotted lines. Hence, **c** and ***u***(p)δu are partly obscured by the semi-opaque “back wall.” The directions of $\boldsymbol{w}_{||}(b)$ and $\boldsymbol{w}_{||}(q)$ were imparted by these rotations.

P. 286. § 29.5.2. This section speaks of constructing a small loop L as a parallelogram whose edges are built out of two unit vector fields, **u** and **v**. As shown in [29.6], [29.7], [29.8], and Figure 1s the parallelogram has sides $\boldsymbol{u}(o)\ \delta$u, **u**(p)$\delta$u, $\boldsymbol{v}(a)\ \delta$v, and **v**(o)$\delta$v, where o, p, and a are arguments for the vectors as well as anchor points. The tiny factors δu and δv provide a way to shrink the vectors so that they ultimately become indistinguishable from segments of the curve L and lie in it. They are absent in the final form of the commutator, which appears on p. 288 as

$$[\boldsymbol{v},\ \boldsymbol{u}] \equiv \nabla_{\boldsymbol{v}}\, \boldsymbol{u} - \nabla_{\boldsymbol{u}}\, \boldsymbol{v}$$

and the Riemann curvature tensor, which appears as (29.8)

$$\boldsymbol{R}(\boldsymbol{u},\ \boldsymbol{v};\ \boldsymbol{w}) \equiv \boldsymbol{\mathcal{R}}(\boldsymbol{u},\ \boldsymbol{v})\ \boldsymbol{w} = [\nabla_{\boldsymbol{u}},\ \nabla_{\boldsymbol{v}}] - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}$$

P. 289, § 29.5.3. At end of ¶ 2, there is an equation:

$$-\mathcal{R}_U(\boldsymbol{\epsilon}) = \delta_{\boldsymbol{\epsilon}}\left(\angle \mathbf{U}_{||}\ \mathbf{U}\right) \asymp \vec{\nabla}_{\boldsymbol{\epsilon}}\left(\angle \mathbf{U}_{||}\ \mathbf{U}\right) = \mid \nabla_{\boldsymbol{\epsilon}}\ \mathbf{U} \mid$$

Judging from the use of symbols on previous pages, the $\mathcal{R}$ is real, while the emboldened $\boldsymbol{\epsilon}$, ∇, and **U** are shown as vectors. The font I am using now lacks a bold nabla so I added the superscripted arrow. If this is correct, a real first term is equated to a real second term, an apparent vector third term, and a real fourth term. Since $\mathcal{R}_U$ refers to the holonomy on a 2-surface, perhaps all the terms should be real. Perhaps $\delta_{\boldsymbol{\epsilon}}$ and $\nabla_{\boldsymbol{\epsilon}}$ are real, where $\delta_{\boldsymbol{\epsilon}}$ refers to distance on $\boldsymbol{\epsilon}$ and $\nabla_{\boldsymbol{\epsilon}}$ refers to $\delta_{\boldsymbol{\epsilon}}$ obtained as the gradient. Does one need vertical brackets on the fourth term? They appear to conflict with the minus sign on $\mathcal{R}$. Vasco (P. c. Forum) wrote: *The bold nabla symbol should be a normal nabla symbol which would then give the third term the meaning of the directional derivative of the angle along ϵ, which is a scalar, and, importantly, is also another way of writing the second term*. *It is also easy to see how this could happen when typesetting the book for printing*.

P. 289. § 29.5.3. At the bottom of p. 289, a series of equations represent the holonomy illustrated in [29.8] and ultimately enable the derivation of the Riemann Tensor on p. 290. The derivation is fairly obvious except for the rewriting of the commutator term $\nabla_{\boldsymbol{c}}\ \boldsymbol{w}(b)$ as $\delta \mathrm{u} \delta \mathrm{v} \nabla_{[\boldsymbol{v},\boldsymbol{u}]}\ \boldsymbol{w}(o)$ in line 3. This is to provide the derivation.

$$\delta \mathrm{u} \delta \mathrm{v} \nabla_{[\boldsymbol{v},\boldsymbol{u}]}\ \boldsymbol{w}(o) = \delta \mathrm{u} \delta \mathrm{v} \nabla_{(\nabla_{\boldsymbol{u}}\, \boldsymbol{v} - \nabla_{\boldsymbol{v}}\, \boldsymbol{u})}\ \boldsymbol{w}(o) = \nabla_{(\delta \mathrm{u} \delta \mathrm{v}\, \nabla_{\boldsymbol{u}}\, \boldsymbol{v} - \delta \mathrm{u} \delta \mathrm{v}\, \nabla_{\boldsymbol{v}}\, \boldsymbol{u})}\ \boldsymbol{w}(o) = \nabla_{(\ \nabla_{\boldsymbol{u}\delta \mathrm{u}}\, \boldsymbol{v} \delta \mathrm{v} - \nabla_{\boldsymbol{v}\delta \mathrm{v}}\, \boldsymbol{u} \delta \mathrm{u})}\ \boldsymbol{w}(o)$$

$$= \nabla_{\boldsymbol{c}}\ \boldsymbol{w}(o)\ = \nabla_{\boldsymbol{c}}\ \boldsymbol{w}(b)$$

In line 1 above, everything in the subscript comprises the direction of the little **c** vector that we see in [29.7] and [29.8], essentially the change of **u** on **v** subtracted from the change of **v** on **u**. In the final term of line 1, above, the two terms of the subscript are equivalent to the contents of the two boxes in [27.7]. Here, they are obtained with a simple linear rearrangement. Then the nabla is the change of **w** under parallel-transport on **c**. The final shift of arguments from b to o (taking the derivation in reverse) appears to lack explicit justification as the arguments are hidden by the **[v, u]** notation. The holonomy calculated at point b adds the changes in direction of $\boldsymbol{w}_{||}(b)$, which itself is the cumulative result of changes to $\boldsymbol{w}$ (o), which is the vector undergoing parallel-transport.

P. 291. *Carry out the lengthy (but straightforward) computation that proves that, when evaluated at o,*

$$\boldsymbol{R}\left(\boldsymbol{u},\ \boldsymbol{v};\ \boldsymbol{w}^{NEW}\right) = \boldsymbol{R}\left(\boldsymbol{u},\ \boldsymbol{v};\ \boldsymbol{w}^{OLD}\right)$$

Proof: Needham hints that four terms containing derivatives of f cancel each other out. It is given that $f(0) = 1$ and $\boldsymbol{w}^{\mathrm{OLD}}(0) = \boldsymbol{w}^{\mathrm{NEW}}(0)$. We will use the equivalence between (29.5) and the equation above it.

$$\mathcal{R}\left(\boldsymbol{u},\ \boldsymbol{v};\ f\boldsymbol{w}^{\mathrm{OLD}}\right) = \left\{[\nabla_{\boldsymbol{u}},\ \nabla_{\boldsymbol{v}}] - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\right\}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right)$$

$$\equiv \delta \mathrm{u} \delta \mathrm{v}\left[\{\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}}\}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right) - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right)\right]$$

$$= \delta \mathrm{u} \delta \mathrm{v}\left[\left\{\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right) - \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right)\right\} - \nabla_{(\nabla_{\boldsymbol{u}} \boldsymbol{v} - \nabla_{\boldsymbol{v}} \boldsymbol{u})}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right)\right] \quad \text{(by p. 288, 2nd box)}$$

$$= \delta \mathrm{u} \delta \mathrm{v}\Big[\left\{\nabla_{\boldsymbol{u}}\left(\nabla_{\boldsymbol{v}}(f)\,\boldsymbol{w}^{\mathrm{OLD}} + f\nabla_{\boldsymbol{v}}\left(\boldsymbol{w}^{\mathrm{OLD}}\right)\right) - \left(\nabla_{\boldsymbol{v}}\left(\nabla_{\boldsymbol{u}}(f)\,\boldsymbol{w}^{\mathrm{OLD}} + f\nabla_{\boldsymbol{u}}\left(\boldsymbol{w}^{\mathrm{OLD}}\right)\right)\right)\right\} - \left\{\nabla_{(\nabla_u v - \nabla_v u)}(f)\,\boldsymbol{w}^{\mathrm{OLD}} + f\nabla_{(\nabla_u v - \nabla_v u)}\left(\boldsymbol{w}^{\mathrm{OLD}}\right)\right\}\Big]$$

(Leibniz)

Now we need to apply a second round of Leibniz's product rule to each of the four terms in the first set of curly brackets. These must be the four terms that "miraculously" cancelled each other out, but they do not cancel out completely. We number the terms on the RHS as {1,2}, {3,4}, {5,6}, {7,8}

$$\nabla_{\boldsymbol{u}}\left(\nabla_{\boldsymbol{v}}(f)\,\boldsymbol{w}^{\mathrm{OLD}}\right) = \nabla_{\boldsymbol{u}}(\nabla_{\boldsymbol{v}}(f))\,\boldsymbol{w}^{\mathrm{OLD}} + \nabla_{\boldsymbol{v}}(f)\,\nabla_{\boldsymbol{u}}\,\boldsymbol{w}^{\mathrm{OLD}}$$

$$\nabla_{\boldsymbol{u}}\left(f\nabla_{\boldsymbol{v}}\left(\boldsymbol{w}^{\mathrm{OLD}}\right)\right) = \nabla_{\boldsymbol{u}}(f)\,\nabla_{\boldsymbol{v}}\,\boldsymbol{w}^{\mathrm{OLD}} + \mathrm{f}\nabla_{\boldsymbol{u}}\,\nabla_{\boldsymbol{v}}\,\boldsymbol{w}^{\mathrm{OLD}}$$

$$-\nabla_{\boldsymbol{v}}\left(\nabla_{\boldsymbol{u}}(f)\,\boldsymbol{w}^{\mathrm{OLD}}\right) = -\nabla_{\boldsymbol{v}}(\nabla_{\boldsymbol{u}}(f))\,\boldsymbol{w}^{\mathrm{OLD}} - \nabla_{\boldsymbol{u}}(f)\,\nabla_{\boldsymbol{v}}\,\boldsymbol{w}^{\mathrm{OLD}}$$

$$-\nabla_{\boldsymbol{v}}\left(f\nabla_{\boldsymbol{u}}\left(\boldsymbol{w}^{\mathrm{OLD}}\right)\right) = -\nabla_{\boldsymbol{v}}(f)\,\nabla_{\boldsymbol{u}}\,\boldsymbol{w}^{\mathrm{OLD}} - \mathrm{f}\nabla_{\boldsymbol{v}}\,\nabla_{\boldsymbol{u}}\,\boldsymbol{w}^{\mathrm{OLD}}$$

Then, adding all the terms and recalling that the order of 2nd derivatives does not matter, we find that 5 cancels 1, 7 cancels 2, 6 cancels 3, leaving just the sum of terms 4 and 8

$$\mathrm{f}\nabla_{\boldsymbol{u}}\,\nabla_{\boldsymbol{v}}\,\boldsymbol{w}^{\mathrm{OLD}} - \mathrm{f}\nabla_{\boldsymbol{v}}\,\nabla_{\boldsymbol{u}}\,\boldsymbol{w}^{\mathrm{OLD}}$$

The second derivatives do not cancel each other because they are applied to a vector rather than a scalar. Then, remembering that f(0) = 1, we can write

$$f\,\{\nabla_{\boldsymbol{u}}\,\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\,\nabla_{\boldsymbol{u}}\}\,\boldsymbol{w}^{\mathrm{OLD}} = f[\nabla_{\boldsymbol{u}} - \nabla_{\boldsymbol{v}}]\,\boldsymbol{w}^{\mathrm{OLD}}$$

The commutator term inside the **second set** of curly brackets can be rewritten, again applying Leibniz:

$$\nabla_{(\nabla_{\boldsymbol{u}}\boldsymbol{v}-\nabla_{\boldsymbol{v}}\boldsymbol{u})}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right)=\nabla_{\nabla_{u}\boldsymbol{v}}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right)-\nabla_{\nabla_{\boldsymbol{v}}\boldsymbol{u}}\left(f\boldsymbol{w}^{\mathrm{OLD}}\right)$$

$=\left\{\nabla_{\nabla_{\boldsymbol{u}}\boldsymbol{v}}(f)\,\boldsymbol{w}^{\mathrm{OLD}}+f\nabla_{\nabla_{u}v}\,\boldsymbol{w}^{\mathrm{OLD}}\right\}-\left\{\nabla_{\nabla_{\boldsymbol{v}}\boldsymbol{u}}(f)\,\boldsymbol{w}^{\mathrm{OLD}}+f\nabla_{\nabla_{v}u}\,\boldsymbol{w}^{\mathrm{OLD}}\right\}$ [Note : $\boldsymbol{w}^{\mathrm{OLD}}$ is argument of $\nabla_{\boldsymbol{x}}$ if not preceded by (f).]

I believe the commutator terms with derivatives of f cancel by virtue of equal projections. Cancelling terms, we get

$$\text{commutator}=f\left\{\nabla_{\nabla_{u}v}\,\boldsymbol{w}^{\mathrm{OLD}}-\nabla_{\nabla_{v}u}\,\boldsymbol{w}^{\mathrm{OLD}}\right\}=\mathrm{f}\nabla_{[u,v]}\,\boldsymbol{w}^{\mathrm{OLD}}$$

Then we can write

$$f\left\{[\nabla_{\boldsymbol{u}}-\nabla_{\boldsymbol{v}}]\,\boldsymbol{w}^{\mathrm{OLD}}-\nabla_{[u,v]}\,\boldsymbol{w}^{\mathrm{OLD}}\right\}$$

$$\equiv f\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\,\boldsymbol{w}^{\mathrm{OLD}}(0)=1\cdot\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\,\boldsymbol{w}^{\mathrm{NEW}}(0)=\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\,\boldsymbol{w}^{\mathrm{NEW}}\equiv\boldsymbol{R}\left(\boldsymbol{u},\boldsymbol{v};\boldsymbol{w}^{\mathrm{NEW}}\right)$$

P. 292, §29.5.4, ¶4. *We leave the very short calculation to you, which establishes that*

$$\boldsymbol{R}\,(k_1\,\boldsymbol{u_1}+k_2\,\boldsymbol{u_2},\ \boldsymbol{v};\boldsymbol{w})=k_1\,(\boldsymbol{R}(\boldsymbol{u_1},\ \boldsymbol{v};\boldsymbol{w})+k_2\,(\boldsymbol{R}(\boldsymbol{u_2},\ \boldsymbol{v};\boldsymbol{w}).$$

From p. 288 and p. 290, we have

$$[\boldsymbol{v},\ \boldsymbol{u}]\equiv\nabla_{\boldsymbol{v}}\,\boldsymbol{u}-\nabla_{\boldsymbol{u}}\,\boldsymbol{v}$$

and

$$\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{w})\equiv\mathcal{R}(\boldsymbol{u},\boldsymbol{v})\,\boldsymbol{w}=\left\{[\nabla_{\boldsymbol{u}},\nabla_{\boldsymbol{v}}]-\nabla_{[\boldsymbol{u},\boldsymbol{v}]}\right\}\boldsymbol{w}\qquad\text{[by (29.6)]}$$

$$\boldsymbol{R}\,(k_1\,\boldsymbol{u_1}+k_2\,\boldsymbol{u_2},\boldsymbol{v};\boldsymbol{w})=\left\{(\nabla_{k_1\boldsymbol{u_1}}\boldsymbol{v}+\nabla_{k_2\boldsymbol{u_2}}\boldsymbol{v})-\nabla_{\boldsymbol{v}}(k_1\,\boldsymbol{u_1}+k_2\,\boldsymbol{u_2})-\left(\nabla_{[k_1\boldsymbol{u_1}+k_2\boldsymbol{u_2},\boldsymbol{v}]}\right\}\boldsymbol{w}$$

$$=\left\{\left(k_1\,\nabla_{\boldsymbol{u_1}}\boldsymbol{v}+k_2\,\nabla_{\boldsymbol{u_2}}\boldsymbol{v}-(k_1\,\nabla_{\boldsymbol{v}}\,\boldsymbol{u_1}+k_2\,\nabla_{\boldsymbol{v}}\,u_2)-\left(k_1\,\nabla_{[\boldsymbol{u_1},\boldsymbol{v}]}+k_2\,\nabla_{[\boldsymbol{u_2},\boldsymbol{v}]}\right)\right)\right\}\boldsymbol{w}$$

$$=\left\{(k_1\,\nabla_{\boldsymbol{u_1}}\boldsymbol{v}-k_1\,\nabla_{\boldsymbol{v}}\,\boldsymbol{u_1})-k_1\,\nabla_{[\boldsymbol{u_1},\boldsymbol{v}]}\right\}\boldsymbol{w}+\left\{(k_2\,\nabla_{\boldsymbol{u_2}}\boldsymbol{v}-k_2\,\nabla_{\boldsymbol{v}}\,\boldsymbol{u_2})-k_2\,\nabla_{[\boldsymbol{u_2},\boldsymbol{v}]}\right\}\boldsymbol{w}$$

$$=k_1\left\{(\nabla_{\boldsymbol{u_1}}\boldsymbol{v}-\nabla_{\boldsymbol{v}}\,\boldsymbol{u_1})-\nabla_{[\boldsymbol{u_1},\boldsymbol{v}]}\right\}\boldsymbol{w}+k_2\left\{(\nabla_{\boldsymbol{u_2}}\boldsymbol{v}-\nabla_{\boldsymbol{v}}\,\boldsymbol{u_2})-\nabla_{[\boldsymbol{u_2},\boldsymbol{v}]}\right\}\boldsymbol{w}$$

$$=k_1\left\{[\nabla_{\boldsymbol{u_1}},\nabla_{\boldsymbol{v}}]-\nabla_{[\boldsymbol{u_1},\boldsymbol{v}]}\right\}\boldsymbol{w}+k_2\left\{[\nabla_{\boldsymbol{u_2}},\nabla_{\boldsymbol{v}}]-\nabla_{[\boldsymbol{u_2},\boldsymbol{v}]}\right\}\boldsymbol{w}$$

$$=k_1\,\boldsymbol{R}(\boldsymbol{u_1},\boldsymbol{v};\boldsymbol{w})+k_2\,\boldsymbol{R}(\boldsymbol{u_2},\boldsymbol{v};\boldsymbol{w})$$

Needham did not explain how to expand the subscripted commutator, but the structure shown here with linearly independent commutator segments for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ seems to preserve the linearity of the tensor. From 288, we deduce that

$$[k\boldsymbol{v},\ \boldsymbol{u}] \equiv \nabla_{k\boldsymbol{v}}\, \boldsymbol{u} - \nabla_{\boldsymbol{u}}\, k\boldsymbol{v} = k\nabla_{\boldsymbol{v}}\, \boldsymbol{u} - k\nabla_{\boldsymbol{u}}\, \boldsymbol{v} = k\,(\nabla_{\boldsymbol{v}}\, \boldsymbol{u} - \nabla_{\boldsymbol{u}}\, \boldsymbol{v}) = k[\boldsymbol{v},\ \boldsymbol{u}]$$

P. 300, § 29.6.2. *Geometrical Implications of the Sectional Jacobi Equation (from reply to Vasco)*.

Needham is still using the construction in which $\boldsymbol{\xi}$ connects neighboring geodesics. One could say:

The fiducial geodesic (the central straight line perpendicular to the circle) serves as a neighboring geodesic for all of the particles on the circle. Every plane Π (section of the manifold) contains (1) the fiducial geodesic, (2) the geodesic path of the particle **v**, and (3) the connecting vector $\boldsymbol{\xi}$. The straight line fiducial geodesic can be geodesic only because attention is focused on a single plane in the 3-manifold.

This construction is like that of [28.7] and constructions plotted on the surface, but unlike those, it does not slide a tangent plane along a general surface. Instead, every section becomes a new plane on which curvature is measured, and because the particles are equidistant from the fiducial particle, the curvature at time t can be calculated as an average of the curvatures of any two orthogonal sections (pp. 303-304).

I still find myself puzzled by [29.2]. I understand that each of the planes is a section of a 3-manifold. Each of the three planes has a distinctive geodesic triangle, illustrating three different curvatures. But I still have not been able to imagine a physical model for the figure.

Can we take each of the three axes and rotate one of the triangles about the axis? But then when A2 rotates into A1, what does that mean? Does A2 morph smoothly into A1 and each of those into A2?

Perhaps each plane and geodesic triangle is simply meant to represent a possible section of a 3-manifold, but one that would be incompatible with the other two illustrated planes. In paragraph 2, p. 282, Needham says the three sectional curvatures are all independent. I take that to mean we would treat each of them as independent of the other two, i.e. as separate models of curvatures transposed onto one 3-manifold. Then there is no single physical model which subsumes the figure.

Pp. 301-302. §29.6.3. Computational proof of the Jacobi Equation

We are given in line -6, p. 301, that $\boldsymbol{\mathcal{R}}(\boldsymbol{v},\ \boldsymbol{\xi}) = \left[\nabla_{\boldsymbol{v}},\ \nabla_{\boldsymbol{\xi}}\right]$. Beginning with line -3,

$$\nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} \boldsymbol{\xi} = \nabla_{\boldsymbol{v}} \nabla_{\xi} \boldsymbol{v}$$

$$= \left[\nabla_{\boldsymbol{v}}, \nabla_{\xi}\right] \boldsymbol{v} + \nabla_{\xi} \left(\nabla_{\boldsymbol{v}} \boldsymbol{v}\right)$$

Since $\nabla_{\boldsymbol{v}} \boldsymbol{v} = 0$ one is left with

$$\nabla_{\boldsymbol{v}} \nabla_{\xi} \boldsymbol{v} = \left[\nabla_{\boldsymbol{v}}, \nabla_{\xi}\right] \boldsymbol{v} \qquad (1)$$

and then one can proceed to $\mathcal{R}(\boldsymbol{v}, \boldsymbol{\xi}) = \left[\nabla_{\boldsymbol{v}}, \nabla_{\xi}\right]$, but there is no immediate source for (1). From p. 289, line -1, we can deduce the following:

$-\delta \boldsymbol{w}_{||} = \delta u \delta v(\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} \boldsymbol{w} - \nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} \boldsymbol{w})$, which implies (29.6) that $\mathcal{R}(\boldsymbol{u}, \boldsymbol{v}) = [\nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{v}}] - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}$.

In our case, since the commutator vanishes and **v** corresponds to **w**, it would imply that

$$\left(\nabla_{\boldsymbol{v}} \nabla_{\xi} \boldsymbol{v} - \nabla_{\xi} \nabla_{\boldsymbol{v}} \boldsymbol{v}\right) = \nabla_{\boldsymbol{v}} \nabla_{\xi} \boldsymbol{v} \quad \Longrightarrow \quad \mathcal{R}(\boldsymbol{v}, \boldsymbol{\xi}) = \left[\nabla_{\boldsymbol{v}}, \nabla_{\xi}\right].$$

The second term in parentheses drops out because $\nabla_{\xi} \nabla_{\boldsymbol{v}} \boldsymbol{v} = \nabla_{\xi} 0 = 0$. This proves the Jacobi equation $\nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} \boldsymbol{\xi} = -\mathcal{R}(\boldsymbol{\xi}, \boldsymbol{v}) \boldsymbol{v}$. Note that $\mathcal{R}(\boldsymbol{\xi}, \boldsymbol{v}) = -\mathcal{R}(\boldsymbol{v}, \boldsymbol{\xi})$.

Out[∘]=

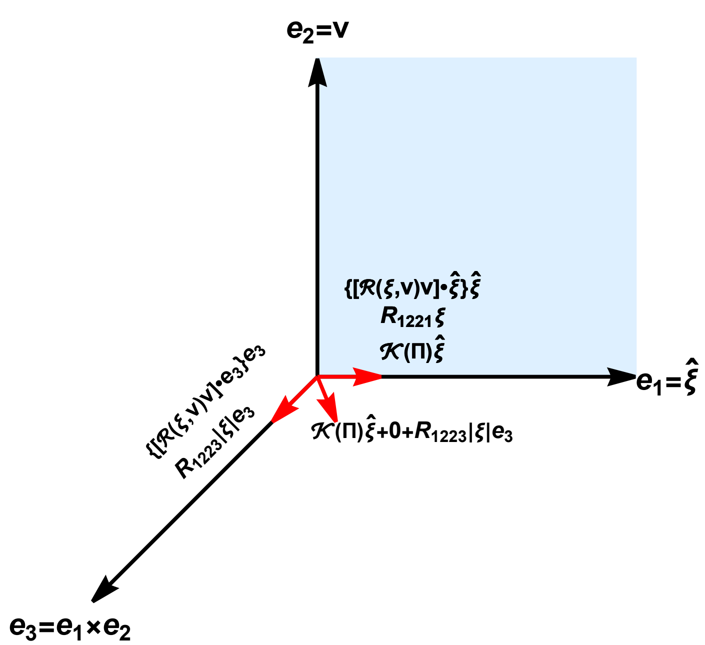

Figure 2. Sectional curvature schematic projected onto e_1 and e_3

P. 302. §29.6. Recover sectional form of Jacobi equation

It is worthwhile to study the equation beginning on line 6, because each of the three terms on the RHS of line 6 represents a parallel transport of the vector **v** over one of the three bases. The three red vectors in Figure 2 represent the resulting holonomy. For the purpose of providing a sectional version of the Jacobi equation, only the first term is needed. The red vector on $\boldsymbol{e_3}$ acts to rotate the section (blue rectangle) as the traversal on $\boldsymbol{e_1}$ progresses. $\mathcal{K}$ and R are real values. $\boldsymbol{\mathcal{R}}$ indicates a vector. The following is an expansion of the equations on lines 6 to 8 to make the derivation more clear. Attention is focused on the first term in the sum on the RHS. The other two terms are as in the text:

$$\mathcal{R}(\boldsymbol{\xi}, \boldsymbol{v})\,\boldsymbol{v} = \left\{[\mathcal{R}(\boldsymbol{\xi}, \boldsymbol{v})\,\boldsymbol{v}] \cdot \hat{\boldsymbol{\xi}}\right\} \hat{\boldsymbol{\xi}} + \{[\mathcal{R}(\boldsymbol{\xi}, \boldsymbol{v})\,\boldsymbol{v}] \cdot \boldsymbol{v}\}\,\boldsymbol{v} + \left\{\left[\mathcal{R}\left(\mid \boldsymbol{\xi} \mid \hat{\boldsymbol{\xi}}, \boldsymbol{v}\right)\boldsymbol{v}\right] \cdot \boldsymbol{e_3}\right\} \boldsymbol{e_3}$$

$$= \left\{\left[\mathcal{R}\left(\mid \boldsymbol{\xi} \mid \hat{\boldsymbol{\xi}}, \boldsymbol{v}\right)\boldsymbol{v}\right] \cdot \hat{\boldsymbol{\xi}}\right\} \hat{\boldsymbol{\xi}} + 0 + \ldots$$

$$= \left\{\left[\left| \boldsymbol{\xi} \right| \mathcal{R}\left(\hat{\boldsymbol{\xi}}, \boldsymbol{v}\right)\boldsymbol{v}\right] \cdot \hat{\boldsymbol{\xi}}\right\} \hat{\boldsymbol{\xi}} + 0 + \ldots \qquad \text{(by the linearity of } \boldsymbol{\mathcal{R}}\text{)}$$

$$= \left\{\left[\mathcal{R}\left(\hat{\boldsymbol{\xi}}, \boldsymbol{v}\right)\boldsymbol{v}\right] \cdot \hat{\boldsymbol{\xi}}\right\} \left| \boldsymbol{\xi} \right| \hat{\boldsymbol{\xi}} + 0 + \ldots \qquad \text{(by rearranging)}$$

$$= \left\{\left[\mathcal{R}\left(\hat{\boldsymbol{\xi}}, \boldsymbol{v}\right)\boldsymbol{v}\right] \cdot \hat{\boldsymbol{\xi}}\right\} \boldsymbol{\xi} + \{[\mathcal{R}(\boldsymbol{e_1}, \boldsymbol{e_2})\,\boldsymbol{e_2}] \cdot \boldsymbol{e_3}\} \left| \boldsymbol{\xi} \right| \boldsymbol{e_3}$$

$$= \mathcal{R}_{1221}\,\boldsymbol{\xi} + \mathcal{R}_{1223} \mid \boldsymbol{\xi} \mid \boldsymbol{e_3}$$

$$= \mathcal{K}(\Pi)\,\boldsymbol{\xi} + \mathcal{R}_{1223} \mid \boldsymbol{\xi} \mid \boldsymbol{e_3} \qquad \text{(from 29.220}$$

P. 302. §29.7.1. Derive (change in area after time t) = $-\mathcal{K}\delta\mathcal{A}(0)\,t^2$ from McLaurin expansion.

It is given that $\delta\dot{\mathcal{A}}(0) = 0$ and $\delta\ddot{\mathcal{A}}(0) = -2\mathcal{K}\delta\mathcal{A}(0)$. Then the expansion is

$$\delta\mathcal{A}(t) - \delta\mathcal{A}(0) = \frac{\delta\dot{\mathcal{A}}(0)\,t}{1!} + \frac{\delta\ddot{\mathcal{A}}(0)\,t^2}{2!} + \ldots \simeq 0 + \frac{-2\mathcal{K}\delta\mathcal{A}(0)\,t^2}{2!} = -\mathcal{K}\delta\mathcal{A}(0)\,t^2 \quad [\text{Because } \delta\ddot{\mathcal{A}}(0) = -2\mathcal{K}\delta\mathcal{A}(0).]$$

which is the change in area after time t. This is just another form of the Jacobi equation.

P. 303, line -1. *where the last term follows from the symmetry [exercise],* $R_{2331} = R_{1332}$.

Given Riemann symmetries, pp. 294-295:

$R_{jik} = -R_{ijk}$ (from (29.7))
$R_{jikm} = -R_{ijkm}$ (from (29.7))
$R_{ijmk} = -R_{ijkm}$ (29.13)

$R_{\text{ijkm}} = R_{\text{kmij}}$ (29.16)

The derivation of the symmetry:

$R_{\text{ijkm}} \to -R_{\text{jikm}} \Longrightarrow R_{1332} \to -R_{3132}$
$R_{\text{ijmk}} = -R_{\text{ijkm}} \Longrightarrow -R_{3132} \to R_{3123}$
$R_{\text{ijkm}} = R_{\text{kmij}} \Longrightarrow R_{3123} \to R_{2331}$

P. 303. Derivation of $\mathcal{K}_{\text{mean}}$ from $\mathcal{K}(\theta) = c^2\,\mathcal{K}(0) + s^2\,\mathcal{K}\left(\frac{\pi}{2}\right) + 2\,\text{scR}_{1332}$. Treat $\mathcal{K}(0)$ and $\mathcal{K}\left(\frac{\pi}{2}\right)$ as constants.

$$\mathcal{K}_{\text{mean}} = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{K}(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left[c^2\,\mathcal{K}(0) + s^2\,\mathcal{K}\left(\frac{\pi}{2}\right) + 2\,\text{scR}_{1332}\right]d\theta$$

$$= \frac{1}{2\pi}\left[\pi\,\mathcal{K}(0) + \pi\,\mathcal{K}\left(\frac{\pi}{2}\right) + 2\cdot 0\cdot R_{1332}\right] = \frac{\mathcal{K}(0)+\mathcal{K}\left(\frac{\pi}{2}\right)}{2}$$

To obtain the equations in the large box on p. 304, Apply $\mathcal{K}_{\text{mean}}$ to the second equation in §29.7.1:

$$(\ddot{\mathcal{A}})(0) = -2\,\mathcal{K}\,\delta\mathcal{A}(0) = -2\,\mathcal{K}_{\text{mean}}\,\delta\mathcal{A}(0) = -2\left[\frac{\mathcal{K}(0)+\mathcal{K}\left(\frac{\pi}{2}\right)}{2}\right]\delta\mathcal{A}(0) = -\left[\mathcal{K}(0) + \mathcal{K}\left(\frac{\pi}{2}\right)\right]\delta\mathcal{A}(0)$$

and to the third equation in §29.7.1:

$$\delta\mathcal{A}(t) - \delta\mathcal{A}(0) \asymp -\mathcal{K}\,\delta\mathcal{A}(0)\,t^2 = -\mathcal{K}_{\text{mean}}\,\delta\mathcal{A}(0)\,t^2 = -\frac{1}{2}\left[\mathcal{K}(0) + \mathcal{K}\left(\frac{\pi}{2}\right)\right]\delta\mathcal{A}(0)\,t^2$$

P. 305, §§29.7.2. *The fact that **R**icci is indeed a tensor in its own right follows immediately **[exercise]** from the fact that **R** is a tensor*.

A tensor is a multilinear operator on vectors.
A linear or multilinear operator on a multilinear operator is still a multilinear operator.
Dot product is a multilinear operator (multiplication and addition).
Σ is a linear operator.
$\boldsymbol{R}\text{icci} = \sum_{m=1}^{n} \boldsymbol{R}(\boldsymbol{e}_m, \boldsymbol{v}; \boldsymbol{w})\bullet\boldsymbol{e}_m$ combines Σ, **R**, and •, so **R**icci is a multilinear operator, and therefore a tensor.

P. 305, ¶1. *It also follows from the symmetries of **R** that **R**icci is symmetric*
$\boldsymbol{R}\text{icci}(w, v) = \boldsymbol{R}\text{icci}(v, w) \iff R_{\text{kj}} = R_{\text{jk}}$.

$R_{\text{mjk}^m} = R_{\text{mjkm}}$ (29.10)

$= R_{\text{kmmj}}$ (Algebraic Bianchi Identity)

$= -R_{\text{mkmj}} = R_{\text{mkjm}} = R_{\text{mkj}^m}$ (1st and 2nd Riemann antisymmetries)

Therefore, $R_{\text{mkj}^m} = R_{\text{mjk}^m} \quad \Longrightarrow \quad R_{\text{kj}} = R_{\text{jk}}$