Initial Comments on OpenAI "Riemann Tensor in 3-Manifolds"

GBP, April 5, 2025. 11 am

P. 1, ¶ 3:

The OpenAI says "The notation $R(e_1, e_2; e_3)$ seems to refer to the curvature tensor evaluated on three vector fields." This suggests that the notation used by Needham is unusual.

P. 1. Special Case in 3-Manifolds

Ricci tensor is introduced. The Reimann tensor is expressed in terms of Ricci. Although I see no difference in typeface, I assume the RHS "R"s are all Ricci, that is Riemann tensors in which the first and last indices are equal and are therefore omitted (See double box at top of P. 305, Needham). I assume that each index number (i.e. 1, 2, 3) represents a specific vector from a set of three basis vectors?

What is g_{ac}? a metric defined on two basis vectors from the same set? Why is there a fourth letter (a, b, c, d) in a 3-manifold? Is the fourth vector the one undergoing parallel-transport? The equation bears only a little similarity to the Riemann curvature tensor presented in paragraph 3.

And what is the equation for the metric? Is this the same metric as that introduce in Needham (30.6), p. 314: $\mathbf{g(u,v)} \equiv \mathbf{u \cdot v}$? This is also written with specific indices as $g_{ij} \equiv \boldsymbol{g}(\boldsymbol{e}_i, \boldsymbol{e}_j)$. I queried OpenAI further on this point and got the following answer:

The metric tensor is a symmetric, bilinear form that, at each point, takes two vectors X and Y in the tangent space and returns a scalar:

$$g\,(X,\ Y) = g_{ab}\, X^a\, Y^b.$$

Here:

- g_{ab} *are the* ***components*** *of the metric tensor in a chosen coordinate system* $(x^1,\ x^2,\ \ldots,\ x^n)$*, where a and b run over the coordinate indices.*
- X^a *and* Y^b *are the* ***components*** *of the vector fields X and Y in this coordinate system.*
- *The Einstein convention is assumed....*

In local coordinates, the metric tensor g_{ab} *describes how the space is stretched or compressed in different directions.*

What does g_{ab} ***represent geometrically****?*

- g_{11} *gives the* ***squared length*** *of the basis vector in the* x^i*-direction.*
- $g_{12} = g_{21}$ *(...symmetric) gives the inner product between the basis vectors in the* x^1 *– and* x^2 *– directions.*
- *If* $g_{ab} = \delta_{ab}$ *(the Kronecker delta), it corresponds to the metric of flat Euclidean space, where vectors*

are orthogonal and have unit length.

***Why is** g_{ab} **important**?*

- *The metric determines the geometry of the manifold, including curvature, distances, and angles.*
- *It allows us to compute lengths of vectors and angles between vectors. For example, the length of a vector X is given by* $\| X \| = \sqrt{g_{ab} X^a X^b}$.
- *It also enables the covariant derivative and curvature tensors to be defined. The covariant derivative depends on how the metric varies from point to point, and this variation is related to the curvature of the manifold.*

Example: Metric Tensor in Polar Coordinates

...

$$ds^2 = g_{11}\, dr^2 + 2 g_{12}\, dr d\theta + g_{22}\, d\theta^2,$$

where:

- g_{11} *is the coefficient of* dr^2,
- g_{12} *(and* g_{21}*) is the coefficient of* $dr d\theta$,
- g_{22} *is the coefficient of* $d\theta^2$.

For the flat Euclidean plane, the metric in polar coordinates is given by: $ds^2 = dr^2 + r^2\, d\theta^2$, *which means:*

$g_{11} = 1$ *(because* dr^2 *just contributes a factor of 1),*

$g_{12} = 0$ *(there is no cross-term involving both dr and dθ),*

$g_{22} = r^2$ *(because* $d\theta^2$ *is scaled by* r^2*).*

***What does this tell us about** g_{12}?*

- *In this case,* $g_{12} = 0$, *which means that in the polar coordinate system, the differential changes in r and θ are independent of each other.*
- *In other words, the change in the radial direction r does not "mix" with the change in the angular direction θ. There is no term that couples the two coordinates in this flat Euclidean geometry.*

Example: Metric Tensor in a Sheared Coordinate System

Now, let's look at a situation where g_{12} *is* ***non-zero***.

Consider a sheared coordinate system where the coordinates are still (x,y)(x,y), but the yy-coordinate is stretched or compressed depending on the xx-coordinate. For simplicity, let's assume that the metric tensor in this new coordinate system is:

$$g_{ab} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix},$$

where α is a constant representing the shear. In this case, we have:

- $g_{11} = 1$,
- $g_{12} = g_{21} = \alpha$,
- $g_{22} = 1$.

What does $g_{12} = \alpha$ represent in this context?

- *The term g_{12} (and its symmetric counterpart g_{21}) indicates that the coordinates x and y are not* ***orthogonal****. There is a* ***shear*** *or* ***mixing*** *between the two coordinates.*

- *This means that an infinitesimal change in the x-direction also affects the y-coordinate, or conversely, a change in y can affect x due to the shear.*

For example, if you have a small movement in the x-direction, it causes a small movement in the y-direction as well, which wouldn't happen in an orthogonal coordinate system (like the polar coordinates where $g_{12} = 0$).

P. 2. ¶ 2: The equation appears to be an outer product which might replace Riemann $[\nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{v}}]$.

It is interesting that the orthonormal frame simplifies to $R(e_1, e_2)\, e_3 = 0$ in S^3.

P. 3. Step 4. OpenAI introduces the non-zero Riemann tensor, but makes no comment on whether the bases are orthonormal. The non-zero Riemann tensor happens where "the Ricci tensor is not proportional to the metric (i.e. the space is not of constant curvature)". I can't visualize or conceptualize this.

P. 4. Step 1. ¶ 3. I don't think I understand what an "orthonormal frame" is. It appears to be much like a set of coordinates. Why is it introduced here?

P. 4. Step 1. ¶ 1. How does one derive a matrix like this?

P. 4. Step 1. ¶ 4. The "dual coframe (one-forms)" is introduced as $\{\theta^1, \theta^2, \theta^3\}$. These must correspond to what Needham writes as $\{\boldsymbol{\omega}^i\}$ or $\{\mathbf{d}x, \mathbf{d}y, \mathbf{d}z\}$. Why are they introduced? Is it because the metric consists of the sum of the squared one-forms (Cf. equation, para. 1)?

P. 4. Step 2. Determine the Levi-Civita connection and find nonzero one-forms.

Given:

$d\theta^i + \omega^i_j \wedge \theta^j = 0$ (First structure equation)

$\theta^1 = dx$, $\theta^2 = dy$, $\theta^3 = dz - x dy$ (From step 1)

Find ω^3_1. Instantiate first structure equation.

$d\theta^3 + \omega^3_1 \wedge \theta^1 = 0$ (Note: $d\theta^3 \equiv d(\theta^3)$, where 3 is the index, not a power.)

$d\theta^3 = d(dz - xdy) = d(dz) - d(xdy) = 0 - \left(\left(\frac{\partial x}{\partial x}\right) dx \wedge dy + x\, d(dy)\right) = -dx \wedge dy$ (note d(dz)=0, d(dy) = 0)

$d\theta^3 + \omega^3_1 \wedge \theta^1 = -dx \wedge dy + \omega^3_1 \wedge dx = -\theta^1 \wedge \theta^2 + \omega^3_1 \wedge \theta^1 = 0 \Longrightarrow \omega^3_1 \wedge \theta^1 = -\theta^2 \wedge \theta^1$

$\Longrightarrow \omega^3_1 = -\theta^2 = -dy$ This is what we wanted.

Find ω^2_3. Instantiate first structure equation.

$d\theta^2 + \omega^2_3 \wedge \theta^3 = 0$

$d\theta^2 = d(dy) = 0$

$d\theta^3 = -dx \wedge dy$ (from previous)

$0 + \omega^2_3 \wedge \theta^3 = 0 \Longrightarrow \omega^2_3 \wedge \theta^3 = \omega^2_3 \wedge (dz - xdy) = 0$

The answer is given as $\omega^2_3 = \theta^1 = dx$. **I do not see a way to derive this.**

These answers are supposed to reveal “how the basis vectors change under parallel transport.” I’m not sure how the answers would show that.

Then OpenAI says “From these, the Christoffel symbols can be determined:” **How does one determine a Christoffel symbol from ω^2_3 or ω^3_1? Does $\nabla_{e_1} e_3 = \omega^3_1$? Is $\nabla_{e_1} e_3$ a Christoffel symbol?** Step 2 seems a bit incoherent. There is no indication of how the Christoffel symbols are determined from ω^3_1 and ω^2_3.

The following equations are presented. I added the last equation, because it is used on p. 5. The term $\nabla_{e_1} e_2$ must equal zero because it amounts to $\frac{\partial\left(\frac{\partial}{\partial y}\right)}{\partial x}$. **How are these equations derived?**

$\nabla_{e_1} e_3 = -e_2, \quad \nabla_{e_3} e_1 = e_2, \nabla_{e_2} e_3 = e_1, \nabla_{e_3} e_2 = -e_1, \quad \nabla_{e_1} e_2 = 0$

From these equations, the Riemann tensor is calculated:

$$R(e_1, e_2)\, e_3 = \nabla_{e_1} e_1 - \nabla_{e_2} (-e_2) - \nabla_{[e_1,e_2]} e_3 = 0 + 0 - \nabla_{[e_1,e_2]} e_3 = 0$$

P. 5. Step 4. Compute

$$R(e_1, e_3)\, e_2 = \nabla_{e_1} (-e_1) - \nabla_{e_3} (0) - \nabla_{[e_1,e_3]} e_2 = 0 - 0 - 0 = 0$$

P. 6. Step 1. How does one go about defining an orthonormal frame (e_1, e_2, e_3)? It looks like the orthonormal frame is

$$\mathbf{e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{f(x)} \frac{\partial}{\partial y}, \quad e_3 = \frac{1}{g(x)} \frac{\partial}{\partial z}}$$

Why is that? And what is it good for? It doesn't seem to be used here.

The dual coframe is $\theta^1 = \mathrm{dx}, \quad \theta^2 = \sinh(x)\, \mathrm{dy}, \quad \theta^3 = e^x\, \mathrm{dz}$.

Step 2: Compute Connection One-Forms

$$\mathrm{d}\theta^2 = d(\sinh(x)\, \mathrm{dy}) = \cosh(x)\, \mathrm{dx} \wedge \mathrm{dy} + 0 \wedge d(\mathrm{dy}) = \cosh(x)\, \mathrm{dx} \wedge \mathrm{dy}$$

$$= \cosh(x)\, \theta^1 \wedge \frac{\theta^2}{\sinh(x)} = \frac{\cosh(x)}{\sinh(x)}\, \theta^1 \wedge \theta^2$$

$$\mathrm{d}\theta^3 = e^x\, \mathrm{dx} \wedge \mathrm{dz} = \mathrm{dx} \wedge e^x\, \mathrm{dz} = \theta^1 \wedge \theta^3$$

Then OpenAI says "From these, we obtain the nonzero connection one-forms:"

$$\omega_1^2 = \frac{\cosh(x)}{\sinh(x)}\, \theta^2, \quad \omega_1^3 = \theta^3$$

We assume they satisfy the structure equation, so we write:

$$\mathrm{d}\theta^2 + \omega_1^2 \wedge \theta^1 = 0$$

$$\frac{\cosh(x)}{\sinh(x)}\, \theta^1 \wedge \theta^2 + \omega_1^2 \wedge \theta^1 = 0$$

$\omega_1^2 \wedge \theta^1 = \frac{\cosh(x)}{\sinh(x)}\, \theta^2 \wedge \theta^1 \implies \omega_1^2 = \frac{\cosh(x)}{\sinh(x)}\, \theta^2$ This is what we wanted.

$$d\theta^3 + \omega^3_1 \wedge \theta^1 = 0$$

$$\theta^1 \wedge \theta^3 + \omega^3_1 \wedge \theta^1 = 0 \quad \Longrightarrow \quad \omega^3_1 \wedge \theta^1 = -\theta^1 \wedge \theta^3 = \theta^3 \wedge \theta^1 \quad \Longrightarrow \quad \omega^3_1 = \theta^3$$

Step 3. Then the Riemann Tensor is to be computed using the second type of structure equation.

$$R^i_j = d\omega^i_k + \omega^i_k \wedge \omega^k_j$$

From Step 2, $\omega^2_1 = \frac{\cosh(x)}{\sinh(x)}\,\theta^2$.

$$d\omega^2_1 = d\left(\frac{\cosh(x)}{\sinh(x)}\,\theta^2\right) = d\left(\frac{\cosh(x)}{\sinh(x)}\right)\theta^2 + \frac{\cosh(x)}{\sinh(x)}\,d\theta^2 \qquad \text{(Leibniz product rule)}$$

$$= \frac{\sinh(x)\sinh(x) - \cosh(x)\cosh(x)}{\sinh^2(x)}\,dx \wedge \theta^2 + \frac{\cosh(x)}{\sinh(x)}\left(\frac{\cosh(x)}{\sinh(x)}\,\theta^1 \wedge \theta^2\right) \quad (d\theta^2 \text{ from p. 6, Step 2})$$

$$= \frac{(1/2)[(\cosh(2x)-1)-(\cosh(2x)+1)]}{\sinh^2(x)}\,\theta^1 \wedge \theta^2 + \frac{\cosh^2(x)}{\sinh^2(x)}\,\theta^1 \wedge \theta^2$$

$$= -\frac{1}{\sinh^2(x)}\,\theta^1 \wedge \theta^2 + \frac{\cosh^2(x)}{\sinh^2(x)}\,\theta^1 \wedge \theta^2$$

$$= \frac{\cosh^2(x)-1}{\sinh^2(x)}\,\theta^1 \wedge \theta^2$$

(**This differs from the third equation on p 7. Why is the second term of the Leibniz product rule not applied by the OpenAI ?** The steps for calculating $d\theta^2$ are given explicitly on p. 13, and those for deriving $d\omega^2_1$ are given explicitly on p. 14, so it is clear that the AI calculation has not carried through the product rule.)

OpenAI computes the Riemann forms $R^2_1 = d\omega^2_1 = -\frac{1}{\sinh^2(x)}\,\theta^1 \wedge \theta^2$ and $R^3_1 = d\omega^3_1 = \theta^1 \wedge \theta^3$, apparently using the value found by faulty use of the Leibniz product rule.

Pp. 8, 9. On pp. 8 and 9 it repeats much of this. Near the bottom of p. 9, we find “Since $\omega^3_1 \wedge \omega^1_3 = 0$, the curvature two-form simplifies to $R^3_1 = \theta^1 \wedge \theta^3$.” How do we know that $\omega^3_1 \wedge \omega^1_3 = 0$? Use the first structure equation where $\theta^3 = e^x\,dz$.

$$d\theta^1 + \omega^1_3 \wedge \theta^3 = 0$$

$$\omega^1_3 \wedge (e^x\,dz) = -d(dx) = 0 \quad \Longrightarrow \quad \omega^1_3 \wedge (e^x\,dz) = 0 \quad \Longrightarrow \quad \omega^1_3 = 0 \quad \Longrightarrow \quad \omega^3_1 \wedge \omega^1_3 = 0$$

At the bottom of p. 9, we find that the curvature two-form $R_1^2 = -\frac{1}{\sinh^2(x)}\,\theta^1 \wedge \theta^2$ corresponds to $R_{1212} = -\frac{1}{\sinh^2(x)}$ and $R_1^3 = \theta^1 \wedge \theta^3$ corresponds to $R_{1313} = 1$.

P. 10. Step 4. Here the OpenAI computes $R(e_1, e_2;\ e_3)$ to demonstrate that the Riemann tensor on the **warped product metric** is non-zero.

It is computing $R(X, Y)\,z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ to obtain $R_{1213} = 0$ and $R_{1323} = 0$

How is this done? We need to find an equivalence between the nabla tensors and the symbols of the one-forms. On p. 7, Steps 3 and 4, we find

$R_1^2 = \mathrm{d}\omega_1^2 = -\frac{1}{\sinh^2(x)}\,\theta^1 \wedge \theta^2$, $\quad R_{1212} = -\frac{1}{\sinh^2(x)} \quad ? = R\big(e_i,\, e_j;\, e_i\big) \bullet \mathrm{e}_j \quad ? = (\nabla_X \nabla_Y - \nabla_Y \nabla_X)\,\mathrm{X} \bullet \mathrm{Y}$

$R_1^3 = \mathrm{d}\omega_1^3 = \theta^1 \wedge \theta^3 \quad R_{1313} = 1 \quad ? = R(e_i,\, e_k;\, e_i) \bullet \mathrm{e}_k \quad ? = (\nabla_X \nabla_Z - \nabla_Z \nabla_X)\,\mathrm{X} \bullet \mathrm{Z}$

$R_{1213} = 0 \quad ? = R\big(e_i,\, e_j;\, e_i\big) \bullet \mathrm{e}_k \quad ? = (\nabla_X \nabla_Y - \nabla_Y \nabla_X)\,\mathrm{X} \bullet \mathrm{Z}$

$R_{1323} = -1 \quad ? = R\big(e_i,\, e_k;\, e_j\big) \bullet \mathrm{e}_k \quad ? = (\nabla_X \nabla_Z - \nabla_Z \nabla_X)\,\mathrm{Y} \bullet \mathrm{Z}$

We are given the equation.

$$R(e_1,\, e_2)\, e_3 = R_{1323}\, e_3 = -e_3$$

But the indices of the notation R_{1323} do not correspond to the use of indices in the similar notation presented by Needham on p. 293. By Needham's notation, one would write:

$R_{1213}\, e_3$ corresponds to $(\nabla_{e_1} e_2 - \nabla_{e_2} e_1)\, e_1 \bullet \mathrm{e}_3$
$R_{1323}\, e_3$ corresponds to $(\nabla_{e_1} e_3 - \nabla_{e_3} e_1)\, e_2 \bullet \mathrm{e}_3$