Notes on Needham, Chapter 28.

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P. 271 Derivation of Jacobi Equation. Understanding this derivation requires observing the difference between real ξ and vector $\boldsymbol{\xi}$. The fourth equation on p. 271 has real values on both sides.

$$\frac{d^2\xi}{dt^2} = -\left[\frac{1}{R^2}\right] R\,\delta\theta \sin\left(\frac{t}{R}\right)$$

Needham states that this equation can be written as the Jacobi equation,

$$\ddot{\boldsymbol{\xi}} = -\mathcal{K}\boldsymbol{\xi}$$

which has vectors on both sides, as signaled by the boldface font. The superscript dots indicate derivatives by time, i.e. $\frac{d}{dt}, \frac{d^2}{dt^2}$. How does one get from the real equation to the vector equation?

Two particles start at the same time at the north pole of a sphere of radius R, moving at equal speed along two meridians (geodesics). $\boldsymbol{\xi}$ is described as a "connecting vector". Hence, it makes a horizontal straight line anchored on one of the particles and terminating in the other, as in [28.3]. In this construction, $\boldsymbol{\xi}$ is horizontal at every point in time as the particles traverse the geodesics. The distance between the particles at any moment in time is described as $|\boldsymbol{\xi}| = \xi$ in the second equation on the page. The first and second derivatives, if they exist, are parallel to $\boldsymbol{\xi}$, though they may point in opposite directions. This is because the constant angle between $\boldsymbol{\xi}$ and the meridians, or between $\boldsymbol{\xi}$ and the equator.

In the second equation on p. 271, we find that $\xi = R\,\delta\theta\,\sin\left(\frac{t}{R}\right)$, so we can rewrite the fourth equation.

$$\frac{d^2\xi}{dt^2} = -\frac{1}{R^2}\,\xi$$

Then, because $\ddot{\boldsymbol{\xi}} \,//\, \boldsymbol{\xi}$ we simply multiply both sides by the unit vector $\hat{\boldsymbol{\xi}}$ and replace $\frac{1}{R^2}$ by $\mathcal{K}$:

$$\frac{d^2\xi}{dt^2}\,\hat{\boldsymbol{\xi}} = -\mathcal{K}\,\xi\,\hat{\boldsymbol{\xi}} \quad \Longrightarrow \quad \ddot{\boldsymbol{\xi}} = -\mathcal{K}\,\boldsymbol{\xi}$$

From another point of view, the unit vector is multiplied by equal magnitudes. I credit Vasco (P. C. Forum, VDGF) with calling my attention to the implications of the fact that the derivatives of $\boldsymbol{\xi}$ are parallel to $\boldsymbol{\xi}$ and horizontal at every point in time.

The above derivation could be accomplished as an exercise in geometry without introducing the unnecessary time parameter or the dot notation from Newtonian physics. The derivatives are taken in terms of r, which is distance on the geodesic.

$$r = \mathrm{R}\phi \implies \phi = \frac{r}{R}$$

$$\rho = R \sin\phi = R \sin\left(\frac{r}{R}\right)$$

$$|\,\boldsymbol{\xi}\,| = \xi = \rho\delta\theta = R\,\delta\theta \sin\left(\frac{r}{R}\right)$$

$$\frac{d\xi}{dr} = \left[\frac{1}{R}\right] R\,\delta\theta \cos\left(\frac{r}{R}\right)$$

$$\frac{d^2\xi}{dr^2} = -\left[\frac{1}{R^2}\right] R\,\delta\theta \sin\left(\frac{r}{R}\right)$$

$$\boldsymbol{\xi}'' = -\mathcal{K}\mathrm{R}\,\delta\theta \sin\left(\frac{r}{R}\right) = -\mathcal{K}\boldsymbol{\xi}$$

P. 273. Section 28.1.3 Negative Curvature: The Pseudosphere.

This equation can also be derived geometrically without the use of the notation from Newtonian physics. Let X, short for $X(\rho)$, be the distance of the particles from the symmetry axis. The derivative is taken in terms of ρ.

$$\left|\,\boldsymbol{\xi}(\rho)\,\right| \doteq X(\rho)\,\delta\theta \doteq \frac{|\boldsymbol{\xi}_0|}{R}\,X(\rho) = \xi = \frac{1}{R}\,\left|\,\boldsymbol{\xi}_0\,\right|\,X(\rho) \qquad \text{(28.3 rewritten with } \rho \text{ for t)}$$

$$\frac{-\mathrm{dX}}{\mathrm{d}\rho} = \frac{X}{R}$$

$$\xi' = \left(\frac{1}{R}\,|\,\boldsymbol{\xi}_0\,|\,X(\rho)\right)' = \frac{1}{R}\,|\,\boldsymbol{\xi}_0\,|\,X' = \frac{1}{R}\,|\,\boldsymbol{\xi}_0\,|\,\left(-\frac{X}{R}\right) = -\frac{1}{R}\,\xi$$

$$\xi'' = -\frac{1}{R}\,\xi' = -\frac{1}{R}\left(-\frac{1}{R}\,\xi\right) = \frac{1}{R^2}\,\xi$$

Then by the same reasoning regarding the parallel derivatives as applied on the derivation for p. 271, one can multiply both sides by the unit vector $\hat{\boldsymbol{\xi}}$.

$$\xi''\,\hat{\boldsymbol{\xi}} = \frac{1}{R^2}\,\xi\,\hat{\boldsymbol{\xi}}$$

$$\boldsymbol{\xi}'' = -\mathcal{K}\boldsymbol{\xi} \qquad (\mathcal{K} \text{ on pseudosphere.})$$

The sign seems correct, but inconsistent with the derivation on the sphere, which has negative sign, but constant positive curvature on the sphere.

P. 277 in § 28.2.2. Relative Acceleration = Holonomy of Velocity.

The equation $\dot{\boldsymbol{\xi}}$ is defined as “relative velocity”, which has “increased from zero to $\delta \boldsymbol{v} \asymp \ddot{\boldsymbol{\xi}}\, \delta$t”. This is a little puzzling at first until one considers that time begins at a horizontal geodesic where $\dot{\boldsymbol{\xi}} = 0$. From (28.1) and the text above it, one could write $\delta \boldsymbol{v} \asymp \frac{d\dot{\boldsymbol{\xi}}}{\mathrm{dt}}\, \delta\mathrm{t} = \ddot{\boldsymbol{\xi}}\, \delta\mathrm{t}$.

In sentence 1 of ¶ 3, $\tilde{\boldsymbol{v}}$ is not defined explicitly and it appears again in the following box as $\tilde{\boldsymbol{v}}(\tilde{b})$. Normally, the tilde signifies an object defined on a surface $\mathcal{S}$ rather than on a map of the surface. Here, the plotted surface evidently serves as both map and surface. Then what is intended in ¶ 4, line 1, by “these two vectors at $\tilde{b}$”? It must be $\boldsymbol{v}_{||}$ and the other vector anchored at $\tilde{b}$, so that other vector must be $\tilde{\boldsymbol{v}}(\tilde{b})$, the white vector anchored at $\tilde{b}$.

P. 278. 28.3 The Circumference and Area of a Small Geodesic Circle.

The title requires some attention. My first thought was that “geodesic circle” would refer to a circle which was itself, somehow, a geodesic curve. On a sphere, that that could only be a meridian or an equator. But what would it be on a general surface? It didn’t seem to fit with my understanding of the construction. The index entry for “geodesic circle” takes one to page 10, a helpful entry which I missed on the first go-round. Needham defines “geodesic circle” with a physical construction involving a string of fixed length pulled taut and rotated about a centre point on the surface. The definition amounts to an intrinsic (within the surface) circle drawn touching the ends of all **geodesic** radii at a fixed distance r from a centre. Since the rays are geodesic, they are also intrinsic. Hence, r designates an intrinsic radius. The circle itself is **not** generally geodesic on the surface. One can see an example plotted on a sphere in [2.4], p. 19. That figure also shows another kind of radius --- the extrinsic (not in the surface) radius orthogonal to the axis of rotation. The extrinsic radius is designated ρ. As either r or ρ are allowed to shrink, the two radii ultimately approach equality.

In ¶ 4 of the section, $\xi(r)$ is defined as a “separation”, which I take to be a linear distance, so ξ(r) is a linear function on an intrinsic linear quantity. In the same sentence, Needham wrote of “the same *geodesic polar coordinates* as before” [emphasis added]. Perhaps he is referring to (t, θ), as in §28.2.1, p. 274, ¶3, but he seems to be substituting r for t in the equation $g(r) \equiv \rho(\theta_0, r)$. This suggests he views this as a problem in geometry rather than physics and time. The name $g(r)$ hints at a geodesic, such as one of those “launched at angle θ_0 and $\theta_0 + \delta\theta$”, but as indicated by the equivalence symbol, it must be a form of the extrinsic radius ρ. I take $\rho(\theta, r)$ to be an Euclidean ray emanating at any θ angle from a point on an axis orthogonal to the surface tangent plane at the centre point. Then $g(r)$ designates only the ρ launched at an initial reference angle θ_0. Since $g(r) \equiv \rho(\theta, r)$, one can write $\xi(r) = g(r)\, \delta\theta$. It appears that as $r \to 0$, the plane Π_ρ, orthogonal to the rotational axis of ρ, approaches the tangent plane Π_T to the surface at the centre point and geodesic rays on the surface become indistinguishable from Euclidean (geodesic) rays in Π_ρ and Π_T.

Needham expands $g(r)$ into a McLaurin series:

$$g(r) = r + \frac{1}{2} g''(0)\, r^2 + \frac{1}{6} g'''(0)\, r^3 + \cdots.$$

One can see that the first term derived from $g'(0)\, r$, so we deduce that $g'(0) = 1$. We are given that $g''(0) = 0$, and it follows that

$$g'''(0) = -\mathcal{K}(0)\, g'(0) = -\mathcal{K}(0)$$

and

$$C(r) = 2\,\pi\, g(r) \;=\; 2\,\pi\,(r \;+\; 0 + \frac{1}{6} g'''(0)\, r^3 + \cdots)$$

$$2\,\pi r \;+ \frac{\pi}{3}\left(-\mathcal{K}(0)\, r^3\right) = 2\,\pi r \;- \frac{\pi}{3}\,\mathcal{K} r^3$$

Rearranging gives

$$-\mathcal{K} \asymp \frac{C(r)-2\,\pi r}{(\pi/3)\, r^3} \;=\; > \mathcal{K} \asymp \frac{3}{\pi}\left[\frac{2\,\pi r - C(r)}{r^3}\right]$$

Integration **[exercise]**:

$$\mathcal{A}(r) = \int C(r)\, dr = \int \left(2\,\pi r \;- \frac{\pi}{3}\,\mathcal{K} r^3\right) dr$$

$$= \left[\pi r^2 - \frac{\pi}{12}\,\mathcal{K} r^4\right]_0^r = \pi r^2 - \frac{\pi}{12}\,\mathcal{K} r^4 - (0 - 0) = \pi r^2 - \frac{\pi}{12}\,\mathcal{K} r^4$$

$$\mathcal{K} = -\frac{\mathcal{A}(r) - \pi r^2}{(\pi/12)\, r^4} = \frac{12}{\pi}\left[\frac{\pi r^2 - \mathcal{A}}{r^4}\right]$$

My understanding and interpretation of the symbols and geometry in this section owes much to discussions with Vasco (P.c. VDGF Forum).

Comparative notes on Chapter 28. These thoughts arrived while trying to do Exercise 7, p. 336. I plan to make an excursion through the models to sort out the notation. In Exercise 7, I think we have to merge the model drawn in [28.4b] with that in [28. 6]. The second, [28. 6] provides the template for proving Minding's Theorem (See last sentence before 28.23.2).

The variable *r* appears early in the chapter in [28.3], p. 271.Though it has a prominent place in the exercise, the variable r does not even appear in the model of the pseudosphere [28.4b] or the general model [28.6], and to add confusion, the circle in [28.6] is shown as $K(\sigma)$, so one has to avoid mixing this

K with the curvature $\mathcal{K}$. From this Needham writes on p. 276 a metric with $\rho(t,\ \theta)$, where ρ apparently subs in for σ in [28.6]. He derives $\mathcal{K} = -\frac{\ddot{\rho}}{\rho}$ and implies that this supports the Jacobi equation $\ddot{\boldsymbol{\xi}} = -\mathcal{K}\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is the shrinking separation between two vertical geodesics launched from 0. I think he is basing this on an unwritten equation $\frac{\ddot{\rho}}{\rho} = \frac{\ddot{\boldsymbol{\xi}}}{\boldsymbol{\xi}}$, and I think it is valid because the separation is between geodesics, so geodesic curvature is 0 and the angles remain unchanged under differentiation. He uses the new equation to expand his statement of Minding's Theorem from surfaces of positive curvature to surfaces of constant curvature.

Then we arrive at 28.3, where r is defined as an intrinsic radius of a small circle $K(r)$, so we could say that r has been substituted for σ in [28.6]. Then $g(r) \equiv \rho(\theta_0,\ r)$ and $\xi(r) = g(r)\ \delta\theta$ show that g and ρ are real magnitudes of radii launched from some arbitrary 0 point on the surface. I would like them to be geodesic, but that seems to be debatable.

That is the context we bring to Exercise 7, where we encounter $g'' = -\mathcal{K}$g. This looks like Minding's Theorem again, except that g is a radius rather than a separation ξ. Then we can proceed as in 28.3, using the MacLaurin series to derive an equation for $g(r)$ and substituting in appropriate values for $\mathcal{K}(0)$ to show that each $\mathcal{K}(0)$ equals the corresponding general $\mathcal{K}$.

Chapter 28 would benefit from a rewrite. Illustrating a variety of models for curvature is a worthy goal, but it needs a more coherence, explanation, comparison and contrast, and explicit definitions of symbols.