Needham, VDGF, Chapter 27, Notes
G. Palmer, April 25, 2024

P. 267. [exercise] Verify that general result in (27.6) is in accord with the special case illustrated in [27.3].

(27.6): *Let $\mathcal{S}$ be a surface with metric $d\hat{s}^2 = A^2\,du^2 + B^2\,dv^2$ and let $\hat{L}$ be a simple loop on $\mathcal{S}$, represented by the simple loop L in the (u,v)-map plane. If in the map plane we define the vector field*

$$\boldsymbol{V} \equiv \begin{pmatrix} (\partial_v A)/B \\ (\partial_u B)/A \end{pmatrix}$$

then the holonomy of $\hat{L}$ on $\mathcal{S}$ is equal to the circulation of $\boldsymbol{V}$ *around L in the map:*

$$\mathcal{R}(\hat{L}) = C_L(\boldsymbol{V})$$

Use the metric with polar coordinates from p. 264: $d\hat{s}^2 = dr^2 + r^2\,d\theta^2$. Then, A = 1, B = r, u = r, v = θ.

$$(\partial_v A)/B = \partial_\theta 1/r = 0/r = 0$$
$$(\partial_u B)/A = \partial_r r/1 = 1/1 = 1$$

Then

$$\boldsymbol{V} \equiv \begin{pmatrix} (\partial_v A)/B \\ -(\partial_u B)/A \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{(sign changes according to direction of traversal}$$

$$\delta\mathcal{R}_{\boldsymbol{U}}(\text{edge}) = \boldsymbol{V}\bullet\begin{pmatrix} \delta r \\ \delta\theta \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}\bullet\begin{pmatrix} \delta r \\ \delta\theta \end{pmatrix} = -\delta\theta$$

$$\mathcal{R}(\hat{L}) = C_L(\boldsymbol{V}) = 0 + (-\delta\theta) + 0 + \delta\theta = 0$$

This is in accord with the special case illustrated in [27.3].

P. 268. Geometric Proof of the Metric Curvature Formula.

The discussion in Chapter 27 alternates discussion of the circulation of vector fields about loops with that of circulation of vector fields about rectangles(§27.2). Using polar coordinates, holonomy $\mathcal{R}(\hat{L})$ about a loop in a flat, ultimately rectangular, surface is set equal to circulation $C_L(V)$ about a rectangular loop in a flat map(§27.3). Finally, holonomy about a loop ($\mathcal{R}(\hat{L})$) on a general surface is set equal to circulation about a rectangular flat map(§27.4). The flat map has orthogonal coordinates u and v. With these, a metric is defined on the surface using $d\,\hat{s}^2 = A^2\,\mathrm{du}^2 + B^2\,\mathrm{dv}^2$. The “hat” on any variable indicates that it belongs to the surface rather than the map. In this context $A = \frac{\partial \hat{s_1}}{\partial u}$, which is the ratio of a small increment on a surface per a movement on u of the map, and similarly, $B = \frac{\partial \hat{s_2}}{\partial v}$ (p. 36). Notice then, that the product AB is an infinitesimal change in area **on the surface** due to infinitesimal changes u and v in the map. AB appears as an expression in the denominator of (27.1) and in (27.4) because curvature is circulation divided by area.

Along with the surface metric and its use of scale factors A and B, the following six equations provide the basis for the proof.

pp. 263-4 $C_R(V) \asymp (\partial_u Q - \partial_v P)\,\delta\mathrm{A}$

(27.3) $\mathrm{curl}\begin{pmatrix} P \\ Q \end{pmatrix} \asymp \frac{C_R(V)}{\delta\mathrm{A}} = \partial_u Q - \partial_v P$

(27.4) $\delta\,\hat{A} = (\mathrm{A}\delta\mathrm{u})\,(\mathrm{B}\delta\mathrm{v}) = (\mathrm{AB})\,\delta\mathrm{A} \asymp (\mathrm{AB})\,\mathrm{dudv}$

(27.6) $\boldsymbol{V} = \begin{pmatrix} \frac{\partial_v A}{B} \\ -\frac{\partial_u B}{A} \end{pmatrix}$

(27.6) $\mathcal{R}(\hat{L}) = C_L(\boldsymbol{V})$

It is given on pp. 263-4 that circulation about an ultimately vanishing rectangle is $C_R(\partial_v) \asymp (\partial_u Q - \partial_v P)\,\delta\mathrm{A}$, where $\boldsymbol{V} = \begin{pmatrix} P(u,\, v) \\ Q(u,\, v) \end{pmatrix}$, a vector field, defined on orthogonal coordinates u and v on a surface, that $\delta\,\hat{A} = (\mathrm{AB})\,\delta\mathrm{u}\delta\mathrm{v}$, and $\delta\mathrm{A} \asymp \mathrm{dudv}$. In (27.6) we are given the corresponding vector field where we can let $P(u,\, v) = \partial_v(A)$ and $Q(u,\, v) = \partial_u(B)$ to obtain $\boldsymbol{V} = \begin{pmatrix} \frac{\partial_v A}{B} \\ -\frac{\partial_u B}{A} \end{pmatrix}$. Then, by appropriate substitution into $C_R(V)$, one can write

$$C_R(\boldsymbol{V}) \asymp \left(-\partial_u\left(\tfrac{\partial_u B}{A}\right) - \partial_v\left(\tfrac{\partial_v A}{B}\right)\right)\delta\mathrm{A}$$

$$= -\left(\partial_u\left(\tfrac{\partial_u B}{A}\right) + \partial_v\left(\tfrac{\partial_v A}{B}\right)\right)\mathrm{dudv} = C_L(\boldsymbol{V})$$

$$\mathcal{K}(\hat{p}) = \lim_{\hat{L}\to\hat{p}} \frac{\mathcal{R}(\hat{L})}{\delta\,\hat{A}} = \frac{C_L(\boldsymbol{V})}{\delta\,\hat{A}} = \frac{1}{\mathrm{AB}}\left[\frac{-\left(\partial_u\left(\frac{\partial_u B}{A}\right)+\partial_v\left(\frac{\partial_v A}{B}\right)\right)\mathrm{dudv}}{\mathrm{dudv}}\right] = -\frac{1}{\mathrm{AB}}\left(\partial_v\left[\frac{\partial_v A}{B}\right] + \partial_u\left[\frac{\partial_u B}{A}\right]\right)$$

and by (27.3), p. 264, this is equivalent to $\frac{1}{\mathrm{AB}}$ {curl($\boldsymbol{V}$)}, where $\mathrm{curl}(V) = \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} \times \boldsymbol{V}$. The application of the partial derivatives with × is analogous to the application of multiplication in the cross product.