Notes on Needham, Visual Differential Geometry and Forms, Chapter 23.
G. Palmer, January 25, 2025

**P. 240, ¶ 3. [exercise]**

(23.1): *To parallel transport a tangent vector* ***w*** *to a surface S along a geodesic G with velocity* ***v****, simply keep the angle between* $\boldsymbol{w}_{||}$ *and* ***v*** *constant as it moves along G.*

*This intrinsically defined method does indeed produce the same parallel-transported vector as the three extrinsic methods above, as follows immediately from* ***[exercise]*** *by combining (22.4) and (22.5).*

*(22.4) If two vectors are parallel transported along a curve on a curved surface, then the angle between them remains constant.* This is followed immediately by *Conversely, if only one of the vectors is known to be parallel-transported, and the angle with the second is kept fixed, then that second vector is necessarily parallel-transported also*.

*(22.5) If G is a geodesic traced at unit speed, launched with initial velocity* ***v****, then the velocity at any future time is obtained by parallel-transporting the initial velocity along G. Conversely, given an arbitrary initial point, and an arbitrary initial velocity* ***v*** *tangent to S, parallel-transport of* ***v*** *along* ***v*** *yields the unique geodesic arising from these initial conditions.*

My note on (22.4). In the first sentence the clause “then the angle between them remains constant” is not well defined. Let us first assume for the sake of simplicity that a vector **w** shares its anchor point with the velocity vector **v** and that both vectors lie in the tangent plane to S at any point p on a curve. Then the angle between them refers to the $\theta$ *angle between* $\boldsymbol{w}_{||}$ *and* ***v*** *in the tangent plane to a surface S at p.*

From (22.4) one can conclude from the second sentence that if the angle between **w** and **v** is kept fixed, and **v** can be shown to be parallel transported, then **w** is also parallel transported. It is given that G is a geodesic. Then from (22.5) it must arise from parallel-transport of **v**. Then by keeping $\theta$ fixed, **w** is parallel-transported along G.

**P. 243, (23.3) and following alternative equation.**

Out[ ]=

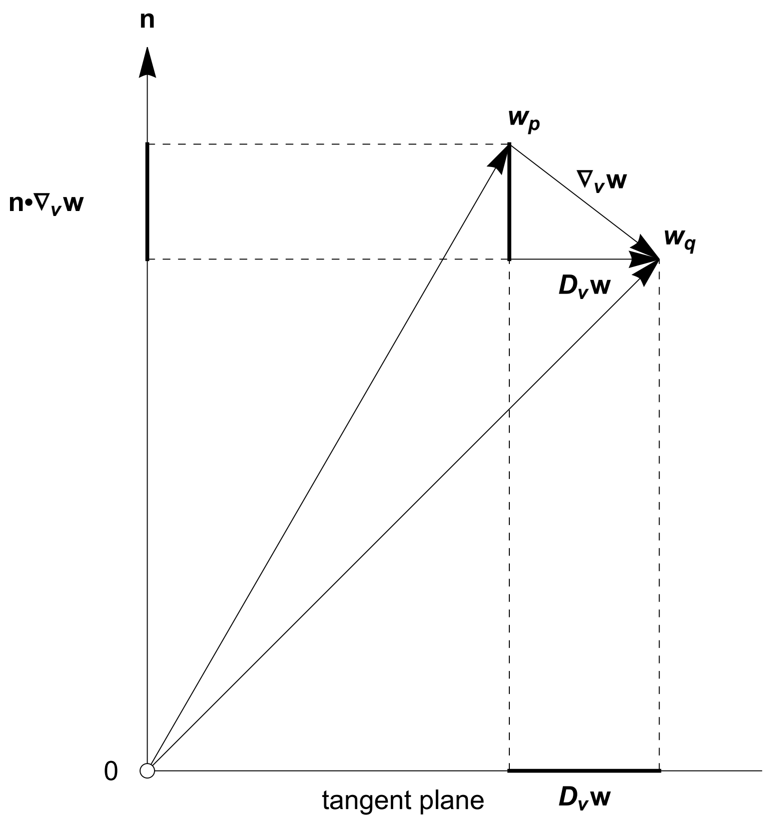


Figure 1. Diagram for (23.3), p. 243

Out[∘]=

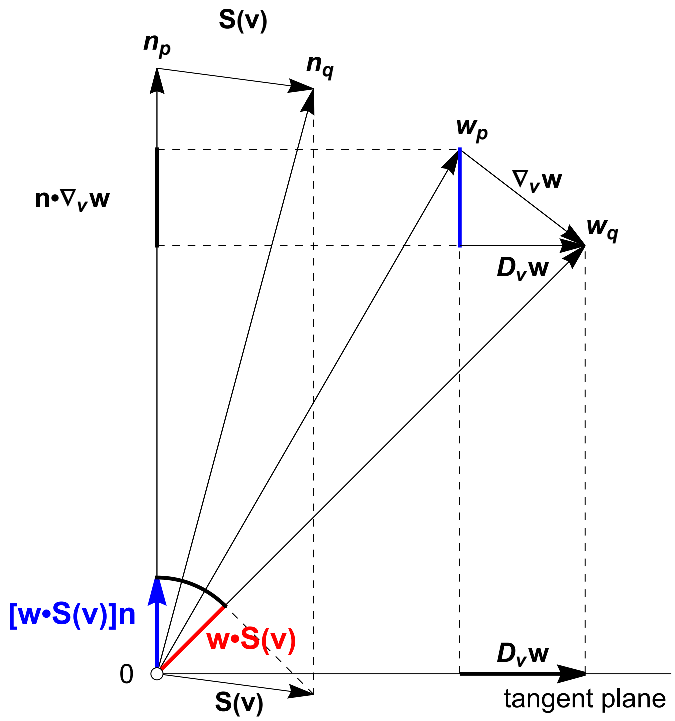


Figure 2. Diagram for (23.3) alternative, p. 243

Note that S(**v**) is projected onto **w** after traversal of p, that is, onto $\boldsymbol{w_q}$. Note also that $\boldsymbol{D_v}$ $\boldsymbol{w}$ is just the projection of $\boldsymbol{w_q} - \boldsymbol{w_p}$ onto the tangent plane after reanchoring to the origin.

**P. 241-243. 23.2 The Intrinsic (aka, “Covariant”) Derivative**

Though I am admittedly lacking experience with the idea of *covariant derivative*, I like the idea of renaming it the *intrinsic derivative*, as anyone who has thought about intrinsic versus extrinsic features would immediately visualize some connection to a tangent plane on a surface.  Barrett O’Neill listed it in the index of *Elementary Geometry* (1997, 2nd Ed., HAP: 325-327) under the sub-heading “intrinsic,” but used “covariant derivative” in the actual discussion. My reading of the section has also turned up a couple of questions:

(1) In [23.3], p. 242, it appears that $\boldsymbol{D_v}$ $\boldsymbol{w}$ is orthogonal to G and v, but it is not so marked, so perhaps it is merely parallel to the vector labeled $\asymp \epsilon \boldsymbol{D_v}\,\boldsymbol{w}$. Or does this perhaps express the idea that the intrinsic derivative vanishes on a geodesic curve (see paragraph 1, p. 242) and paragraph 2, p. 243).

**P. 243, ¶ 5**. The formula (23.3) may be used to prove **[exercise]** that the intrinsic derivative $D_v$ obeys all the same linearity and Leibniz (product) rules as the full vector derivative $\nabla_v$, ...

Given: $D_v\,\boldsymbol{w} = \nabla_v\,\boldsymbol{w} - (\boldsymbol{n}\bullet\nabla_v\,\boldsymbol{w})\,\boldsymbol{n}$

$D_{\boldsymbol{v}}[a\boldsymbol{x} + b\boldsymbol{y}] = \triangledown_{\boldsymbol{v}}[a\boldsymbol{x} + b\boldsymbol{y}] - \boldsymbol{n}\bullet\triangledown_{\boldsymbol{v}}[a\boldsymbol{x} + b\boldsymbol{y}]\,\boldsymbol{n}$

$= \triangledown_{\boldsymbol{v}}[a\boldsymbol{x}] + \triangledown_{\boldsymbol{v}}[b\boldsymbol{y}] - \boldsymbol{n}\bullet(\triangledown_{\boldsymbol{v}}[a\boldsymbol{x}] + \triangledown_{\boldsymbol{v}}[b\boldsymbol{y}])\,\boldsymbol{n}$

$= a(\triangledown_{\boldsymbol{v}}[\boldsymbol{x}] - (\boldsymbol{n}\bullet\triangledown_{\boldsymbol{v}}[\boldsymbol{x}])\,\boldsymbol{n}) + b(\triangledown_{\boldsymbol{v}}[\boldsymbol{y}] - (\boldsymbol{n}\bullet\triangledown_{\boldsymbol{v}}[\boldsymbol{y}])\,\boldsymbol{n})$

$= \mathrm{a}D_{\boldsymbol{v}}\,\boldsymbol{x} + \mathrm{b}D_{\boldsymbol{v}}\,\boldsymbol{y}$

$D_{[f\boldsymbol{x}+g\boldsymbol{y}]}\,\boldsymbol{z} = \triangledown_{[f\boldsymbol{x}+g\boldsymbol{y}]}\,\boldsymbol{z} - \left(\boldsymbol{n}\bullet\triangledown_{[f\boldsymbol{x}+g\boldsymbol{y}]}\,\boldsymbol{z}\right)\boldsymbol{n}$

$= \triangledown_{f\boldsymbol{x}}\,\boldsymbol{z} - (\boldsymbol{n}\bullet\triangledown_{f\boldsymbol{x}}\,\boldsymbol{z})\,\boldsymbol{n} + \triangledown_{g\boldsymbol{y}}\,\boldsymbol{z} - \left(\boldsymbol{n}\bullet\triangledown_{g\boldsymbol{y}}\,\boldsymbol{z}\right)\boldsymbol{n}$

$= \mathrm{f}\triangledown_{\boldsymbol{x}}\,\boldsymbol{z} - (\boldsymbol{n}\bullet\mathrm{f}\triangledown_{\boldsymbol{x}}\,\boldsymbol{z})\,\boldsymbol{n} + \mathrm{g}\triangledown_{\boldsymbol{y}}\,\boldsymbol{z} - \left(\boldsymbol{n}\bullet\mathrm{g}\triangledown_{\boldsymbol{y}}\,\boldsymbol{z}\right)\boldsymbol{n}$

$= f(\triangledown_{\boldsymbol{x}}\,\boldsymbol{z} - (\boldsymbol{n}\bullet\triangledown_{\boldsymbol{x}}\,\boldsymbol{z})\,\boldsymbol{n}) + g\left(\triangledown_{\boldsymbol{y}}\,\boldsymbol{z} - \left(\boldsymbol{n}\bullet\triangledown_{\boldsymbol{y}}\,\boldsymbol{z}\right)\boldsymbol{n}\right)$

$= f\,D_{\boldsymbol{x}}\,\boldsymbol{z} + g\,D_{\boldsymbol{y}}\,\boldsymbol{z}$

$D_{\boldsymbol{v}}[f\boldsymbol{x}] = \triangledown_{\boldsymbol{v}}[f\boldsymbol{x}] - (\boldsymbol{n}\bullet\triangledown_{\boldsymbol{v}}[f\boldsymbol{x}])\,\boldsymbol{n}$

$= \triangledown_{\boldsymbol{v}}[f]\,\boldsymbol{x} + \mathrm{f}\triangledown_{\boldsymbol{v}}[\boldsymbol{x}] - (\boldsymbol{n}\bullet[\triangledown_{\boldsymbol{v}}[f]\,\boldsymbol{x} + \mathrm{f}\triangledown_{\boldsymbol{v}}[\boldsymbol{x}]])\,\boldsymbol{n}$ [by Leibniz]

$= \triangledown_{\boldsymbol{v}}[f]\,\boldsymbol{x} + \mathrm{f}\triangledown_{\boldsymbol{v}}[\boldsymbol{x}] - ([\boldsymbol{n}\bullet\triangledown_{\boldsymbol{v}}[f]]\,\boldsymbol{x} + [\boldsymbol{n}\bullet\mathrm{f}\triangledown_{\boldsymbol{v}}[\boldsymbol{x}]])\,\boldsymbol{n}$

$= (\mathrm{f}\triangledown_{\boldsymbol{v}}[\boldsymbol{x}] - \boldsymbol{n}\bullet\mathrm{f}\triangledown_{\boldsymbol{v}}[\boldsymbol{x}]\,\boldsymbol{n}) + (\triangledown_{\boldsymbol{v}}[f]\,\boldsymbol{x} - (\boldsymbol{n}\bullet[\triangledown_{\boldsymbol{v}}[f]\,\boldsymbol{x})\,\boldsymbol{n})$ [by rearranging]

$= \mathrm{f}D_{v}\,\boldsymbol{x} + f\,'\,\boldsymbol{x}$

$D_{\boldsymbol{v}}[\boldsymbol{x}\bullet\boldsymbol{y}] = \triangledown_{\boldsymbol{v}}[\boldsymbol{x}\bullet\boldsymbol{y}] - (\boldsymbol{n}\bullet\triangledown_{\boldsymbol{v}}[\boldsymbol{x}\bullet\boldsymbol{y}])\,\boldsymbol{n}$

$= \triangledown_{\boldsymbol{v}}[\boldsymbol{x}]\bullet\boldsymbol{y} + \boldsymbol{x}\bullet\triangledown_{\boldsymbol{v}}[\boldsymbol{y}] - \boldsymbol{n}\bullet(\triangledown_{\boldsymbol{v}}[\boldsymbol{x}]\bullet\boldsymbol{y} + \boldsymbol{x}\bullet\triangledown_{\boldsymbol{v}}[\boldsymbol{y}])\,\boldsymbol{n}$ [by Leibniz]

$= \{\triangledown_{\boldsymbol{v}}[\boldsymbol{x}]\bullet\boldsymbol{y} - \boldsymbol{n}\bullet(\triangledown_{\boldsymbol{v}}[\boldsymbol{x}]\bullet\boldsymbol{y})\,\boldsymbol{n}\} + \{\boldsymbol{x}\bullet\triangledown_{\boldsymbol{v}}[\boldsymbol{y}] - \boldsymbol{n}\bullet(\boldsymbol{x}\bullet\triangledown_{\boldsymbol{v}}[\boldsymbol{y}])\,\boldsymbol{n}\}$ [by rearranging]

$= D_{\boldsymbol{v}}[\boldsymbol{x}]\bullet\boldsymbol{y} + \boldsymbol{x}\bullet D_{\boldsymbol{v}}[\boldsymbol{y}]$