Needham, VDGAF, Chapter 21, Exercise 9, p. 220-221.
Palmer, January 25, 2022, 3:15 pm PT

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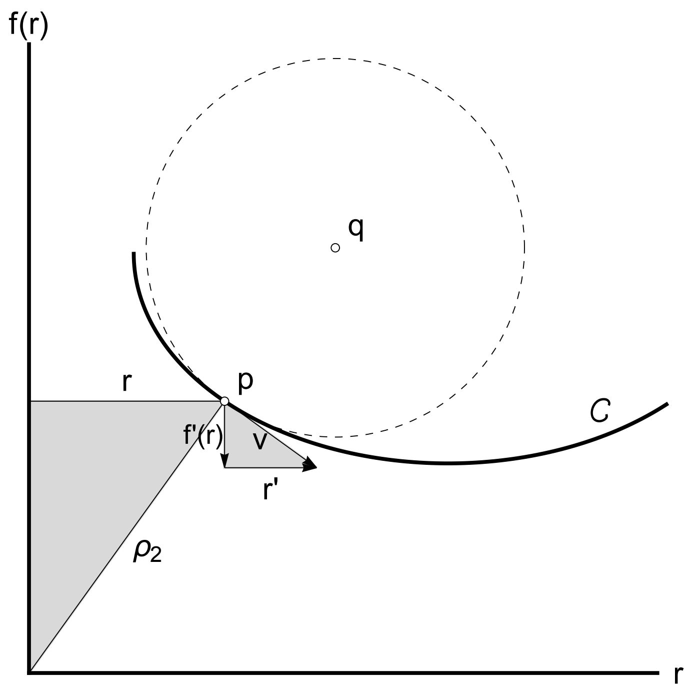

Figure 1. Curve revolving about f(r) axis

9. ***Polar Formula for the Curvature of a Surface of Revolution****. ...describe such a surface by giving its height above a plane perpendicular to its axis of symmetry, which we now take to be the z-axis. If r is the usual polar coordinate in the (x,y)-plane, consider the surface of revolution z = f(r).*

1. By simply changing names appropriately, and using the chain rule, show that (20.1) becomes

$$\mathcal{K}(r) = \frac{f'(r)\, f''(r)}{r\left\{1+(f'(r))^2\right\}^2}$$

(20.1) is $\mathcal{K} = -\frac{y''}{y\left[1+(y')^2\right]^2}$

In Exercise 8, equation (20.1) was derived from [8.6], p. 103, and [10.6], p. 114, by changing from unit speed to variable speed v. So the answer to this exercise also assumes variable speed.

Substitute z = f(r) for y in (20.1) is not sufficient:

$$\mathcal{K} = -\frac{f''}{f\left[1+(f')^2\right]^2}$$

This is not what we wanted. Apparently, one must proceed from a derivation that includes vectors. We return to Exercise 8 and substitute $y \to f(r)$, $x \to r$, as in Figure 1. Then,

$$\kappa_1 = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{v^3} \to \kappa_1 = \frac{\dot{r}\,\ddot{f} - \dot{f}\,\ddot{r}}{v^3} \text{ [using the chain rule on } \frac{d}{dt}\left(\tan\phi = \frac{\dot{f}}{\dot{r}}\right)\text{]}$$

where $v = \sqrt{\dot{r}^2 + \dot{f}^2}$.

Then from Figure 1,

$$\frac{r}{\rho_2} = \frac{\dot{f}}{v} \Longrightarrow \kappa_2 = \frac{\dot{f}}{\mathrm{rv}}$$

and

$$\mathcal{K} = \kappa_1\,\kappa_2 = \frac{\dot{r}\,\ddot{f} - \dot{f}\,\ddot{r}}{v^3}\left(\frac{\dot{f}}{\mathrm{rv}}\right) = \frac{\dot{f}\left(\dot{r}\,\ddot{f} - \dot{f}\,\ddot{r}\right)}{\mathrm{rv}^4}$$

Since the analogy is to exercise 8, we are free to adjust the speed along the curve, so that r represents time: r(t) = t, in which case f(r(t)) = f(r) and the time derivative becomes the r-derivative: $\dot{f} = f\,'$. Since $r(t) = t$, it follows that $\dot{r} = 1$ and $\ddot{r} = 0$. Then substituting to two levels,

$$v = \sqrt{\dot{r}^2 + \dot{f}^2} = \sqrt{1 + (f\,')^2}$$

$$\mathcal{K} = \frac{\dot{f}\left(\dot{r}\,\ddot{f} - \dot{f}\,\ddot{r}\right)}{\mathrm{rv}^4} = \frac{f'\,f''}{r\left\{1+(f')^2\right\}^2}$$

Vasco (P. c., VDGF), has shown how to do the derivation without the trigonometry. This is my own attempt using what I think is his method, though he may think about it differently. Begin with the recognition that z takes the role of x (the axis of rotation) and r takes the role of y (distance from axis of rotation). Then we can write

$z = f(r)$ (A strict parallel would be r = f(z), but that leads to complications.)

$$z\,' = \frac{dz}{dr} = f\,'(r)$$

$$y\,' = \frac{dy}{dx} \Longrightarrow \frac{dr}{dz} = \frac{1}{\mathrm{dz}/\mathrm{dr}} = \frac{1}{f\,'(r)}$$

$$y'' = \frac{d^2y}{dx^2} \Longrightarrow \frac{d^2r}{dz^2} = \frac{d}{dr}\left(\frac{1}{f'(r)}\right)\frac{dr}{dz} = \frac{d}{dr}\left(\frac{1}{f'(r)}\right)\frac{1}{f'(r)} = -\frac{f''(r)}{(f'(r))^2}\cdot\frac{1}{f'(r)}$$

$$y\{1+(y')^2\}^2 \Longrightarrow r\left\{1+\frac{1}{(f'(r))^2}\right\}^2$$

Then substitute into (20.1) for y, y', and y''.

$$\mathcal{K} = -\frac{y''}{y\left[1+(y')^2\right]^2} \Longrightarrow$$

$$\mathcal{K} = \frac{-\frac{f''(r)}{(f'(r))^2}\cdot\frac{1}{f'(r)}}{r\left\{1+\frac{1}{(f'(r))^2}\right\}^2} = -\frac{-\frac{f''(r)}{(f'(r))^2}\cdot\frac{1}{f'(r)}}{r\left\{\frac{(f'(r))^2+1}{(f'(r))^2}\right\}^2} = \frac{\frac{f''(r)}{(f'(r))^2}\cdot\frac{1}{f'(r)}\,(f'(r))^4}{r\left\{1+(f'(r))^2\right\}^2} = \frac{f'(r)\,f''(r)}{r\left\{1+(f'(r))^2\right\}^2}$$

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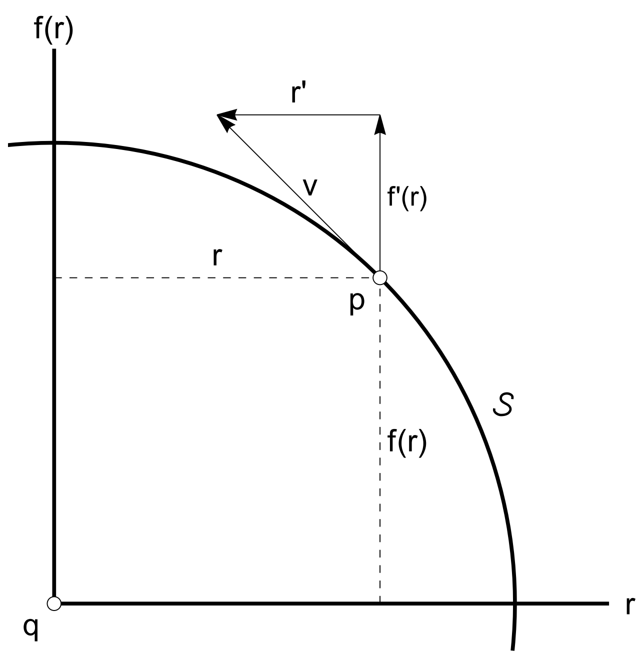

Figure 2. Geodesic line segment on sphere of radius R revolving about f(r) axis

(ii) Find f(r) for a sphere of radius R centred at the origin. Check that the formula in (i) gives the correct value of $\mathcal{K}$*.*

$$R^2 = r^2 + f(r)^2$$

$f(r)^2 = R^2 - r^2$

$\boldsymbol{f(r) = \pm\sqrt{R^2 - r^2}}$

In the following, $f = f(r)$

$f'(r) = -r\Big/\sqrt{R^2 - r^2} = -r/f$

$f''(r) = -\left(R^2\Big/\left(R^2 - r^2\right)^{3/2} = -R^2\Big/f^3\right.$

$f' f'' = (-r/f)\left(-R^2\Big/f^3\right) = \mathrm{rR}^2\Big/f^4$

$(f')^2 = r^2\Big/f^2$

$$\mathcal{K} = \frac{f' f''}{r\left\{1+(f')^2\right\}^2} = \frac{\mathrm{rR}^2/f^4}{r\left\{1+r^2/f^2\right\}^2} = \frac{R^2/f^4}{\left\{1+r^2/f^2\right\}^2} = \frac{R^2/f^4}{\left\{\left(f^2+r^2\right)/f^2\right\}^2}$$

$$= \frac{R^2/f^4}{\left\{R^2/f^2\right\}^2} = \frac{R^2/f^4}{R^4/f^4} = \frac{1}{R^2}$$

This is the value of $\mathcal{K}$ of the sphere given on pages 134 and 142.

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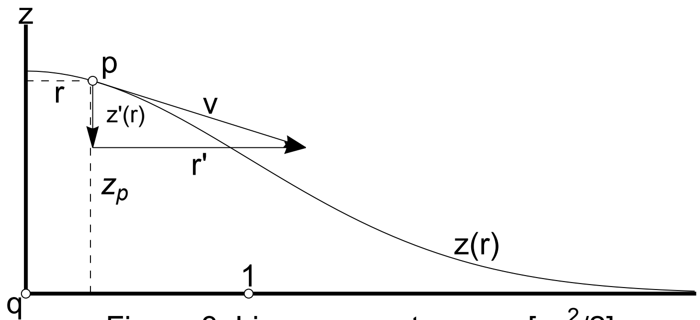

Figure 3. Line segment z = exp[$-r^2$/2]
revolving about z axis

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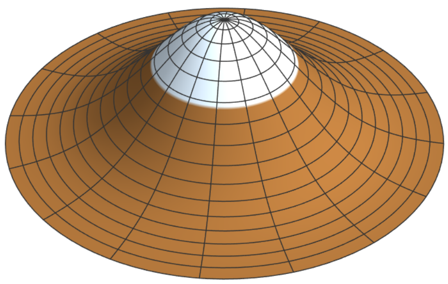

Figure 4. Surface of revolution z = Exp[$-r^2$/2]
$\mathcal{K}$=0 at color boundary

(iii) Manually sketch the surface z = exp[−r ^ 2/2], calculate $\mathcal{K}$, and find the regions of positive, negative, and zero curvature. (A computer-generated graph of the surface should visually confirm your answers.)

$$f' = z' = -r\exp\left[-r^2 \backslash 2\right] = -rz$$

$$f'' = z'' = \left(r^2 - 1\right) z;$$

$$f' f'' = \left(r - r^3\right) z^2$$

$$\mathcal{K} = \frac{f' f''}{r\left\{1+(f')^2\right\}^2} = \frac{\left(r-r^3\right) z^2}{r\left(1+r^2 z^2\right)^2} = \frac{\left(1-r^2\right) z^2}{\left(1+r^2 z^2\right)^2}$$

I imagine the surface as a hill with a rounded top sloping downward into a saddle that opens upwards and encircles the z axis (Figure 4). It is the shape of a mushroom cap that curves upward to a rounded top. By inspection of the curve z (Figure 3), one can see that there are regions of both negative and positive curvature on the geodesic curve z with the inflection point being at or near $r = 1$, where $z = 1/\text{Sqrt}[e] = 0.606531$. This is where $\mathcal{K} = 0$.

The equation finds $\mathcal{K} = 0.710063$ for the concave that opens towards the z-axis at p, where r = .3, and negative $\mathcal{K} = -0.00989468$ at r = 1.5 on the concave away from the z-axis. Under revolution, this segment becomes a saddle. The two curvatures are quite different on the surface of rotation: the region of positive curvature is a like a tire or the outer curved region of a torus sliced through the vertical center plane (cylinder) along the length of the tube; the region of negative curvature is like a saddle. The calculated values of $\mathcal{K}$ agree with Needham, p. 114: positive curvature (concave towards axis of revolution); negative curvature (concave away from the axis of revolution).

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k(r) = (1-r^2) z[r]^2 / (1+r^2 z[r]^2)^2
r = 0.3 = Re[p], z = 0.955997, 𝒦 = 0.710063
r = 1.0, z = 0.606531, 𝒦 = 0.
r = 2.5, z = 0.0439369, 𝒦 = -0.00989468
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