Needham, VDGAF, Chapter 21, Exercise 8, p. 220-221.
Palmer, January 5, 2022, 5:40 pm PT

***8. Variable-Speed Formulas for the Curvature of a Surface of Revolution***. *In the text, we pictured a particle moving at constant unit speed along the generating curve of a surface of revolution, in which the principle curvatures are given by (10.11) and (10.12). If the speed v(t) is not held constant, recall that (10.11) becomes (8.6):*

$$\kappa_1 = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{v^3}$$

*(i) By modifying [10.6], page 114, deduce that the second principle curvature is given by*

$$\kappa_2 = -\frac{\dot{x}}{yv}$$

Modify [10.6] by changing the length of the hypotenuse of the large gray triangle from 1 to $v$. Then, by the similarity of the triangles,

$$\left(\frac{y(t)}{\rho_2} = \frac{\dot{x}}{v}\right) \Longrightarrow \left(\kappa_2\, y(t) = \frac{\dot{x}}{v}\right) \Longrightarrow \left(\kappa_2 = \frac{\dot{x}}{yv}\right)$$

By [10.6], +***n*** is the assigned direction of op, which is an instance of $\rho_1$. Then $\kappa_2$ has the direction of **–*n***, so we write:

$$\kappa_2 = -\frac{\dot{x}}{yv}$$

Regarding (10.13), the phrases “concave towards L” and “concave away from L” are ambiguous. I interpret them to mean that $\mathcal{S}$ concave “towards L” is analogous to $\cap$ with underline L. I am not sure how one might use them to write the proper sign for $\kappa_2$.

*(ii) Deduce that*

$$\mathcal{K} = -\frac{\dot{x}[\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}]}{y\left[\dot{x}^2 + \dot{y}^2\right]^2}$$

Recall that $v = \sqrt{\dot{x}^2 + \dot{y}^2}$

$$\mathcal{K} = \kappa_1\,\kappa_2 = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{v^3} \cdot -\frac{\dot{x}}{yv} = -\frac{\dot{x}[\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}]}{yv^4} = -\frac{\dot{x}[\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}]}{y\left[\dot{x}^2 + \dot{y}^2\right]^2}$$

*(iii) If the particle only moves to the right, never retracing any x-value, then we are free to adjust the speed along the curve so that x represents time: $x(t) = t$, in which case $y(t) = y(x)$, and the time derivative*

*becomes the x-derivative $\dot{y} = y'$. Show that the curvature formula in (ii) then reduces to*

$$\mathcal{K} = -\frac{y''}{y\left[1+(y')^2\right]^2}$$

Since $x(t) = t$, $\dot{x} = 1$ and $\ddot{x} = 0$. Then we can substitute for $\dot{x}$, $\ddot{x}$ and $\dot{y}$:

$$\mathcal{K} = -\frac{\dot{x}[\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}]}{y\left[\dot{x}^2+\dot{y}^2\right]^2} = -\frac{\ddot{y}}{y\left[1+(\dot{y})^2\right]^2} = -\frac{y''}{y\left[1+(y')^2\right]^2}$$