Needham, VDGAF, Chapter 21, Exercise 7, p. 220

Palmer, June 24, 2023, 1:26 pm PST

7. Series Expansion of Surfaces*. For each of the following equations, use a computer to draw the surface. Use series expansions to calculate the quadratic approximation near the origin, then use (10.6), page 111, to deduce the principle curvatures and* $\mathcal{K}$ *there. Visually confirm that your calculations are at least qualitatively correct.*

Out[•]=

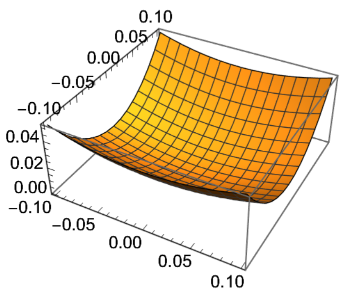

Figure 1. $z = \text{Exp}(x^2 + 4y^2) - 1$

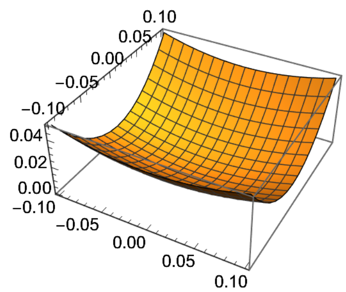

Figure 2. Taylor's series approximation $z = x^2 + 4y^2$

(i) $z = exp(x^2 + 4y^2) - 1.$

This is an equation of the form $f(x,\ y)$, where $f(0,\ 0) = 0$, as discussed in section 10.2. A series expansion for e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

Substitute $(x^2 + 4y^2)$ for x. Then

$$e^{(x^2+4y^2)} = 1 + (x^2 + 4y^2) + \frac{(x^2+4y^2)^2}{2} + \frac{(x^2+4y^2)^3}{6} \cdots$$

$$z = (x^2 + 4y^2) + \frac{(x^2+4y^2)^2}{2} + \frac{(x^2+4y^2)^3}{6} \cdots$$

for which the quadratic form is $z = (x^2 + 4y^2)$. Then use (10.6), which is $z \asymp (1/2)\kappa_1 x^2 + (1/2) k_2 y^2$. Since the coefficients of x and y are 1 and 4, respectively, $\kappa_1 = 2$, $\kappa_2 = 8$. Then $\mathcal{K} = \kappa_1 \kappa_2 = 8$.

We first tried this problem using the Taylor’s series. Wikipedia has an example format for the Taylor’s series with multiple variables. In the following $\boldsymbol{a} \in \boldsymbol{R}^n$.

For example, the third-order Taylor polynomial of a smooth function f: $\mathbf{R}^2 \to \mathbf{R}$ is, denoting $\boldsymbol{x} - \boldsymbol{a} = \boldsymbol{v}$,

$$P_3(\boldsymbol{x}) = f(\boldsymbol{a}) + \frac{\partial f}{\partial x_1}(\boldsymbol{a})v_1 + \frac{\partial f}{\partial x_2}(\boldsymbol{a})v_2 + \frac{\partial^2 f}{\partial x_1^2}(\boldsymbol{a})\frac{v_1^2}{2!} + \frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{a})v_1 v_2 + \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{a})\frac{v_2^2}{2!}$$
$$+ \frac{\partial^3 f}{\partial x_1^3}(\boldsymbol{a})\frac{v_1^3}{3!} + \frac{\partial^3 f}{\partial x_1^2 \partial x_2}(\boldsymbol{a})\frac{v_1^2 v_2}{2!} + \frac{\partial^3 f}{\partial x_1 \partial x_2^2}(\boldsymbol{a})\frac{v_1 v_2^2}{2!} + \frac{\partial^3 f}{\partial x_2^3}(\boldsymbol{a})\frac{v_2^3}{3!}$$

<https://en.wikipedia.org/wiki/Taylor%27s_theorem#Taylor's_theorem_for_multivariate_functions>.

For an approximation near the origin, $(x_0, y_0) = (0, 0)$. Calculate the function and it's derivatives up to f'' at (0, 0). Note that for (0, 0), $e^{x^2+4y^2} = e^0 = 1$.

$$f(0, 0) = \exp(0) - 1 = 1 - 1 = 0$$

$$\frac{\partial f_0}{\partial x} = \frac{\partial\left[\exp\left(x^2+4y^2\right)-1\right]}{\partial x} = 2\,e^{x^2+4y^2}\,x = 2\,x = 0$$

$$\frac{\partial f_0}{\partial y} = \frac{\partial\left[\exp\left(x^2+4y^2\right)-1\right]}{\partial y} = 8\,e^{x^2+4y^2}\,y = 0$$

$$\frac{\partial^2 f_0}{\partial x^2} = 2\,e^{x^2+4y^2} + 4\,x^2\,e^{x^2+4y^2} = 2$$

$$\frac{\partial^2 f_0}{\partial y^2} = 8\,e^{x^2+4y^2} + 64\,y^2\,e^{x^2+4y^2} = 8$$

$$\frac{\partial^2 f_0}{\partial x\,\partial y} = 16\,\mathrm{xy}\,e^{x^2+4y^2} = 0$$

Only the second order terms with x^2 and y^2 have non-zero derivatives.

$$z = f(x, y) \approx \frac{1}{2} \times \left(2\,x^2 + 8\,y^2\right) + \cdots = x^2 + 4\,y^2 + \cdots$$

This answer agrees with (10.3), p. 111, where a = 1, b = 4, and c = 0.

Let κ_x be the principle curvature on the x axis. By Newton's curvature formula (8.3), with the derivatives evaluated at (0,0),

$$\kappa_1 = \kappa_x = \frac{f_x''}{\left\{1+\left(f_x'\right)^2\right\}^{3/2}} = \frac{2}{1^{3/2}} = +2$$

We know the sign is positive because the curve x^2 opens upwards from 0.

$$\kappa_2 = \kappa_y = \frac{8}{1^{3/2}} = +8$$

By (10.6), $z = \frac{1}{2} \times 2\, x^2 + \frac{1}{2} \times 8\, y^2 = x^2 + 4\, y^2$, which approximates the Taylor's series near (0,0). This is to be expected, because on the x-axis, $\theta = 0$ and on the y-axis, $\theta = \pi/2$. But Needham must have intended that we work back from the result for the Taylor's series: $z = x^2 + 4\, y^2 = \frac{1}{2}\, \kappa_1\, x^2 + \frac{1}{2}\, \kappa_2\, y^2 \Longrightarrow \kappa_1 = 2,\ \ \kappa_2 = 8$.

The Gaussian curvature on a surface is the product of the two principle curvatures. $\mathcal{K} = \kappa_1\, \kappa_2$. In this case, $\mathcal{K} = 2 \cdot 8 = 16$. The fact that the surface in Figure 2 opens upwards is apparently not significant, as $\mathcal{K}$ is positive, as one expects of a concave/convex surface.

Out[∘]=

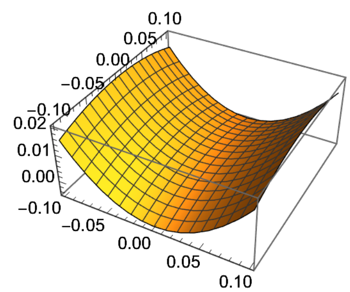

Figure 3. $z = \ln\cos y - \ln\cos 2\, x$

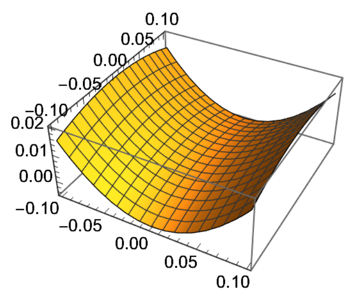

Figure 4. Taylor's series approximation $z = 2\, x^2 - y^2/2$

(ii) $z = \ln\cos y - \ln\cos 2\, x$

A series expansion for z is the sum of the series expansions for ln cos y and (- ln cos 2x).

$z = \ln\cos y - \ln\cos 2\, x = \ln(\cos y / \cos 2\, x)$

A series expansion for cos x is

$$\cos x\ =\ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \simeq 1 - \frac{x^2}{2} \quad \text{(stopping with the quadratic)}$$

So cos y $\simeq 1 - \frac{y^2}{2}$

and $\cos 2\, x\ \simeq 1 - \frac{4\, x^2}{2} = 1 - 2\, x^2$

A series expansion for ln x is

$$\ln x = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} \cdots\right), \text{ where } x \in [-1\ldots1]$$

Then

$$\ln \cos y = \ln\left(1 - \frac{y^2}{2}\right) \asymp -\left(1 - \frac{y^2}{2} \cdots\right) \quad \text{[stopping with the quadratic]}$$

$$\ln \cos 2\,x = \ln\left(1 - \frac{4\,x^2}{2} \cdots\right) \asymp -\left(1 - 2\,x^2\right)$$

$$\ln \cos y \;-\; \ln \cos 2\,x \asymp \left(\frac{y^2}{2} - 1\right) - \left(2\,x^2 - 1\right) = \frac{y^2}{2} - 2\,x^2$$

From (10.6) we can reason that $\kappa_1 = -4$ (on x axis) and $\kappa_2 = 1$ (on y axis). Then $\mathcal{K} = \kappa_1\,\kappa_2 = -4$. Figures 3 and 4 show the expected saddle shape of the negative curvature.

We first tried using the Taylor's series for the cosine and logarithm:

$$f(0,\, 0) = 0$$

$$\frac{\partial f_0}{\partial y} = \frac{\partial[\ln \cos y - \ln \cos 2\,x]}{\partial y} = -\tan\,y\; = 0$$

$$\frac{\partial f_0}{\partial x} = \frac{\partial[\ln \cos y - \ln \cos 2\,x]}{\partial x} = 2 \tan 2\,x = 0$$

$$\frac{\partial^2 f_0}{\partial y^2} = -\sec^2 y = -1$$

$$\frac{\partial^2 f_0}{\partial x^2} = 4 \sec^2 2\,x = 4$$

$$\frac{\partial^2 f_0}{\partial x\,\partial y} = 0$$

Again, only the second order terms with x^2 and y^2 have non-zero derivatives.

$$z = f(x,\, y) \approx \frac{1}{2} \times \left(4\,x^2 - y^2\right) + \cdots = 2\,x^2 - \frac{y^2}{2} + \cdots$$

Working back from this result: $z = 2\,x^2 - \frac{y^2}{2} = \frac{1}{2}\,\kappa_1\,x^2 + \frac{1}{2}\,\kappa_2\,y^2 \Longrightarrow \kappa_1 = 4,\;\; \kappa_2 = -1$. Then $\mathcal{K} = \kappa_1\,\kappa_2 = -4$. The final result is essentially the same, except that the signs of κ_1 and κ_2 are reversed. This result will be used in exercise 17, Chapter 20.