Needham, VDGAF, Chapter 21, Exercise 5, p. 220
Palmer, January 5, 2022, 6 pm PT

*6. The Frenet-Serret Approximation to a Curve. Let **x**(t) be the position at time t of a particle that traces a curve C in space at unit speed. Let ($\boldsymbol{T}_0$, $\boldsymbol{N}_0$, $\boldsymbol{B}_0$) be the Frenet-Serret frame at time t = 0, and let k_0 and τ_0 be the curvature and torsion at this time.*

*(i) Using Taylor's Theorem and the Frenet-Serret equations (9.3), prove that the motion along C is initially given by the **Frenet Approximation**:*

$$\boldsymbol{x}(t) \asymp \boldsymbol{x}(0) + t\boldsymbol{T}_0 + \kappa_0 \frac{t^2}{2} \boldsymbol{N}_0 + \kappa_0 \tau_0 \frac{t^3}{6} \boldsymbol{B}_0 + \cdots$$

The use of "**x**(t)" for motion along *C* seems odd at first, because "x" usually refers to values on the x-axis. But, since boldface "**x**(t)" is a vector, it must go where the curve goes, which we assume to be in 3-space.

From (9.3) $\boldsymbol{T}' = \kappa \boldsymbol{N}$ and $\boldsymbol{N}' = -\kappa \boldsymbol{T} + \tau \boldsymbol{B}$. Let f(x) = **x**(t). Then, we can write

$$f(a) = \boldsymbol{x}(0)$$

$$\frac{f'(a)}{1!}(x - a) = \frac{\boldsymbol{x}'(0)}{1!}(t - 0) = t\boldsymbol{T}_0$$

The Mathematica editor fails to preserve formats in the higher derivatives. Proceeding as above, the third and fourth terms are:

$$\frac{t^2}{2!} \kappa_0 \boldsymbol{N}_0$$

$$-\kappa_0{}^2 t^3 \boldsymbol{T}_0 + \kappa_0 \tau_0 t^3 \boldsymbol{B}_0$$

Since we are considering only infinitesimal increments of time, the term $-\kappa_0^2 t^3 \boldsymbol{T}_0$ is insignificant compared to $t\boldsymbol{T}_0$, so it can be omitted. Then one can write:

$$\boldsymbol{x}(t) \asymp \boldsymbol{x}(0) + t\boldsymbol{T}_0 + \kappa_0 \frac{t^2}{2} \boldsymbol{N}_0 + \kappa_0 \tau_0 \frac{t^3}{6} \boldsymbol{B}_0 + \cdots$$

(ii) Explain the first three terms geometrically.

The first three terms are:

(1) the position of a point **x**(t) at $t = 0$,

(2) velocity: the change in position of **x**(t) at time t in the direction of the tangent at $t = 0$ (and perpendicular to the radius of curvature at $t = 0$), which is influenced by the curvature. It is a change

within the **TN** plane.

(3) acceleration: the change in position of **x**(t) at t in the direction of the normal at $t = 0$. it is a change within the **TN** plane.

(iii) What does the fourth term describe geometrically?

The change in position of of **x**(t) at time t in the direction of the binormal at $t = 0$, in which curvature and the torsion are factors. It is change within the **BN** plane.

(iv) In light of your answer to (iv), why must this term vanish if $\tau_0 = 0$?

Because torsion is a factor in the term. If $\tau_0 = 0$, the third derivative $\tau_0\, \boldsymbol{B}_0 = 0$, and the change in $\boldsymbol{B}_0$ is 0.

(v) Why does it make sense that the fourth term should also vanish if $\kappa_0 = 0$.

If $\kappa_0 = 0$, $\boldsymbol{x}(0)$ is a straight line segment, so there can be no second derivative and consequently, no third derivative term in the Taylor's series, such as that represented by $\kappa_0\, \tau_0\, \frac{t^3}{6}\, \boldsymbol{B}_0$. A straight line does not veer out of a plane in which it lies.