Needham, VDGAF, Chapter 21, Exercise 5, p. 220

Palmer, April 6, 2023, 10:10 am PDT

5. The Helix*. Consider the path of a particle whose position at time t is* ($R\cos\omega t$, $R\sin\omega t$, qt).

(i) *Explain why this is a helix, and state the geometrical/physical interpretations of R, ω, and q.*

This is a helix because the the elements in the parentheses can be assigned to the X, Y, and Z coordinates of 3-dimensional Cartesian space. The X and Y coordinates represent a circle that a point on the curve projects onto the XY-plane even as the Z coordinate causes it to elevate and trace a helix. R is the radius of that circle, or equivalently, of a cylinder on which the helix is traced. ω and q are the angular velocity and height of a point on the helix curve at time 1. That is, $\omega = d\theta/dt$, so $\theta = \omega t$ by analogy to linear velocity, $d = vt$. ω is measured on the horizonal plane.

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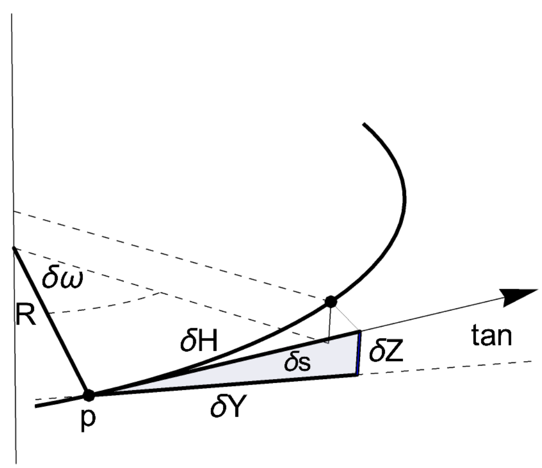

Figure 1. Segment of helix

(ii) Prove that $v = \sqrt{(R\omega)^2 + q^2}$*, and explain this geometrically/physically.*

Let H be the helix. In Figure 1, the arc $\omega\delta t$ lies in the XY plane. The light gray triangle lies in the tangent plane of H. From the triangle, we can write a metric:

$$\delta s^2 = \delta Y^2 + \delta Z^2$$

One can see from the figure that $\delta Z = q\delta t$, and as $\delta t \longrightarrow 0$, $\delta Y \asymp R\delta\omega = R\omega\delta t$ and $\delta H \asymp ds$. Then,

$$\delta H^2 \approx \delta s^2 \approx (R\omega\delta t)^2 + (q\delta t)^2$$

$$\delta H \approx \sqrt{(R\omega)^2 + q^2}\ \delta t$$

$$v = dH/dt \approx \delta H/\delta t \approx \sqrt{(R\omega)^2 + q^2}$$

which we can agree to write as

$$v = \sqrt{(R\omega)^2 + q^2}$$

Geometrically, the particle has an instantaneous velocity $\delta s/dt$ in the direction of the tangent to H, and this component is the root of the squares of horizontal and vertical components δX and δY, both of which lie in the plane tangent to the helix at p, the position of the particle.

(iii) Use the previous question [i.e. exercise] *to calculate κ and τ.*

$$\boldsymbol{v} = \frac{dH}{dt} = (-R\,\omega \sin\omega t,\ R\,\omega \cos\omega t,\ q)$$

$$\dot{\boldsymbol{v}} = \left(-R\,\omega^2 \cos\omega t,\ -R\,\omega^2 \sin\omega t,\ 0\right)$$

$$\boldsymbol{v} \times \dot{\boldsymbol{v}} = (-R\,\omega \sin\omega t,\ R\,\omega \cos\omega t,\ q)\ x\left(-R\,\omega^2 \cos\omega t,\ -R\,\omega^2 \sin\omega t,\ 0\right)$$

$$= \left(q\,R\,\omega^2 \sin\omega t,\ -q\,R\,w^2 \cos\omega t,\ R^2\,w^3\right)$$

$$|\ \boldsymbol{v} \times \dot{\boldsymbol{v}}\ | = |\ \left(q\,R\,\omega^2 \sin\omega t,\ -q\,R\,w^2 \cos\omega t,\ R^2\,w^3\right)| = +\sqrt{R^4\,\omega^6 + q^2\,R^2\,\omega^4\left(\sin^2\omega t + \cos^2\omega t\right)}$$

$$= +\sqrt{R^4\,\omega^6 + q^2\,R^2\,\omega^4} = +\sqrt{R^2\,\omega^4\left(R^2\,\omega^2 + q^2\right)} = R\omega^2\ |\ v\ |$$

$$\kappa = \left|\ \frac{\boldsymbol{v}\times\dot{\boldsymbol{v}}}{v^3}\ \right| = \frac{R\omega^2 |v|}{|v^3|} = \frac{R\omega^2}{|v^2|} = \frac{R\omega^2}{R^2\,\omega^2 + q^2}$$

When $R = 1$, $\kappa < 1$, which seems correct, because the distance on H spanned by angle ω would be greater than the span of the same angle on a unit circle, so the bending with respect to a center of curvature would be less per unit distance traversed.

Find τ

$$\tau = \frac{(\boldsymbol{v}\times\dot{\boldsymbol{v}})\bullet\ddot{\boldsymbol{v}}}{|\boldsymbol{v}\times\dot{\boldsymbol{v}}|^2}$$

We can calculate $\ddot{\boldsymbol{v}}$ and use the value of $\boldsymbol{v}\times\dot{\boldsymbol{v}}$ from previous work.

$$\ddot{\boldsymbol{v}} = d\,\dot{\boldsymbol{v}}/\mathrm{dt} = d\left(-R\,\omega^2\cos\omega\mathrm{t},\ -R\,\omega^2\sin\omega\mathrm{t},\ 0\right)/\mathrm{dt}$$

$$= \left(R\,\omega^3\sin\omega\mathrm{t},\ -R\,\omega^3\cos\omega\mathrm{t},\ 0\right)$$

$$(\boldsymbol{v}\times\dot{\boldsymbol{v}})\bullet\ddot{\boldsymbol{v}} = \left(q\,R\,\omega^2\sin\omega t,\ \ -q\,R\,w^2\cos\omega t,\ \ R^2\,w^3\right)\bullet\left(R\,\omega^3\sin\omega t,\ -R\,\omega^3\cos\omega t,\ 0\right)$$

$$= \mathrm{qR}^2\,\omega^5\sin^2\omega\mathrm{t} + \mathrm{qR}^2\,\omega^5\cos^2\omega\mathrm{t} = \mathrm{qR}^2\,\omega^5\left(\sin^2\omega\mathrm{t} + \cos^2\omega\mathrm{t}\right) = \mathrm{qR}^2\,\omega^5$$

$$|\ v\times\dot{v}\ |^2 = \left(\mathrm{R}\omega^2\ |\ v\ |\right)^2$$

$$\tau = \frac{\mathrm{qR}^2\,\omega^5}{R^2\,\omega^4\,v^2} = \frac{\mathrm{q}\omega}{v^2} = \frac{\mathrm{q}\omega}{R^2\,\omega^2 + q^2}$$

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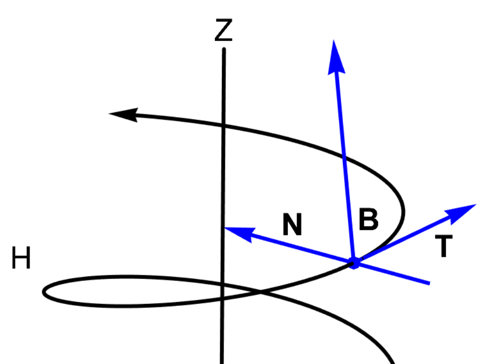

Figure 2. T, N and B on helix H

(iv) Explain ***geometrically*** *why* $lim_{\omega \to \infty}\, \kappa = (1/R)$, *and* $lim_{q \to \infty}\, \kappa = 0$ [should be $lim_{q \to \infty}\, \tau = 0$].

$lim_{\omega \to \infty}\, \kappa = (1/R)$: For a circle $\kappa = (1/R)$. For the helix, $\kappa < (1/R)$ for the same reason I mentioned in part (iv): The distance on H spanned by angle ω in the XY plane would be greater than the span of the same angle on a circle of the same radius, as one can see in Figure 1 by comparing δH or δs to δY, so the bending with respect to a (vertically progressing) center of curvature would be less per unit distance traversed than if the distance were traversed on an inscribed circle. Since a circle has $\kappa = (1/R)$, the helix inscribed in a cylinder of the same radius has $\kappa < (1/R)$ and $lim_{\omega \to \infty}\, \kappa = (1/R)$.

$lim_{q \to \infty}\, \tau = 0$: As $q \to \infty$, the angle between H and a tangent to a latitudinal circle at any point on H remains constant and the curvature remains constant. N is directed towards the centre Z axis. N is horizontal, orthogonal to Z, H, and T. Since N is horizontal, it does not appear to rotate about T. Hence, B cannot rotate about T. Intuitively, I want to believe that $\tau = 0$ for all τ, so one could still write $lim_{q \to \infty}\, \tau = 0$. But it's not that easy; constant $\tau = 0$ conflicts with the calculation in (iii) that $\tau = \frac{q\omega}{R^2\, \omega^2 + q^2}$. So one must reason that N must also undergo some torsion. Consider the Frenet-Serret formula $dN/ds = -\kappa T + \tau B$. From the term τB, one can deduce that N undergoes rotation about T equivalent to the rotation of B about T.

(v) Use (iii) to confirm the predictions of (iv).

$$\kappa = \frac{R\omega^2}{R^2\, \omega^2 + q^2}$$

Try a couple of examples:

Let R = 2, $\omega = 10^6$ and $q = 1$. Then κ should approximate to 1/R.

(1/R)=(1/2)

$\kappa = 2 * 10^{12} / \left(4 * 10^{12} + 1\right) \cong 1/2$, so $\kappa \asymp$ (1/R) as $\omega \to \infty$.

Let R = 2, $\omega = 2\,\pi$, and $q = 10^6$.

$$\tau = \frac{q\omega}{v^2} = \frac{q\omega}{R^2\, \omega^2 + q^2} = \frac{4\,\pi}{4 * 16\,\pi^2 + 10^{12}} \cong \frac{1}{10^{12}}, \text{ so } \tau \asymp 0 \text{ as } q \to \infty.$$

Provide a more general confirmation:

$$lim_{\omega \to \infty}\, \kappa = lim_{\omega \to \infty} \frac{R\omega^2}{R^2\, \omega^2 + q^2} \asymp lim_{\omega \to \infty} \frac{R\omega^2}{R^2\, \omega^2} = \frac{1}{R}$$

$$lim_{q\to\infty}\,\tau = lim_{q\to\infty}\,\frac{q\omega}{R^2\,\omega^2+q^2} \asymp lim_{q\to\infty}\,\frac{q}{q^2} \asymp lim_{q\to\infty}\,\frac{1}{q} \asymp 0$$