Needham, VDGAF, Chapter 20, Exercise 4, p. 219.
Palmer, December 3, 2022, 4:40 pm PT

*4. **Variable Speed Frenet-Serret Equations**. The Frenet-Serret equations (9.3) assume the derivative is with respect to arc length, which is only the same as the time derivative if the curve is traced at unit speed. Suppose instead that the particle has variable speed $v = \dot{s}$.*

(i) Show that $[\Omega] \mapsto v[\Omega]$*:*

From p. 108, $[\Omega] = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$ and

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}' = [\Omega] \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}$$

Let $\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}^{\bullet}$ stand for $d\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}'/dt$. Note that $\boldsymbol{X}' = d\boldsymbol{X}/ds$ and $v = ds/dt = \dot{s}$.

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}^{\bullet} = \begin{pmatrix} \frac{d\boldsymbol{T}'}{dt} \\ \frac{d\boldsymbol{N}'}{dt} \\ \frac{d\boldsymbol{B}'}{dt} \end{pmatrix} = \begin{pmatrix} \frac{d\boldsymbol{T}}{ds}\frac{ds}{dt} \\ \frac{d\boldsymbol{N}}{ds}\frac{ds}{dt} \\ \frac{d\boldsymbol{B}}{ds}\frac{ds}{dt} \end{pmatrix} = \begin{pmatrix} v\boldsymbol{T}' \\ v\boldsymbol{N}' \\ v\boldsymbol{B}' \end{pmatrix} = v[\Omega] \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}$$

This is what we wanted.

(ii) Prove that the acceleration is $\dot{\boldsymbol{v}} = [v\boldsymbol{T}]^{\bullet} = \dot{v}\,\boldsymbol{T} + kv^2\,\boldsymbol{N}$, and interpret and explain both terms geometrically.

Notice the distinction between $\dot{v}$, a real valued derivative dv/dt, and boldface $\dot{\boldsymbol{v}}$, a vector derivative $d\boldsymbol{v}/dt$. We will need $\dot{\boldsymbol{T}}$, which we can find in two ways: From (i),

$$\dot{\boldsymbol{T}} = v[\Omega]_1 \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} = v\,[0 \;\; \kappa \;\; 0] \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} \equiv vk\boldsymbol{N}$$

A more obvious way is

$$\dot{\boldsymbol{T}} = \frac{dT}{dt} = \frac{dT}{ds} \bullet \frac{ds}{dt} = v\boldsymbol{T}' = vk\boldsymbol{N}$$

Then

$$\dot{\boldsymbol{v}} \equiv [v\boldsymbol{T}]^{\bullet} = \left[\dot{v}\,\boldsymbol{T} + v\,\dot{\boldsymbol{T}}\right] \equiv \dot{v}\,\boldsymbol{T} + v(\mathrm{vk}\boldsymbol{N})$$

$$= \dot{v}\,\boldsymbol{T} + \mathrm{kv}^2\,\boldsymbol{N}$$

which is what we wanted.

The first term is instantaneous change of velocity (acceleration) along the **T** axis of the Frenet frame. It is the rate of change of the length or distance along the curve. If the curvature is 0, $\dot{\boldsymbol{v}}$ reduces to $\dot{v}\,\boldsymbol{T}$. Otherwise, the second term is instantaneous of velocity squared in the direction of the **N** axis. It is acceleration in the direction of **N** in the **TN** plane.

(iii) Show that $\boldsymbol{B} = \frac{\boldsymbol{v}\times\dot{\boldsymbol{v}}}{|\boldsymbol{v}\times\dot{\boldsymbol{v}}|}$.

One does not need to calculate the answer. The direction of **v** is the same as **T**. Therefore, the direction of $\dot{\boldsymbol{v}}$ is instantaneously the same as **N**. Hence, the direction of the cross product $\boldsymbol{v}\times\dot{\boldsymbol{v}}$ is instantaneously the same as **B**. The denominator makes the quotient a unit vector. Hence, it must be **B**. Nevertheless, it is good to show it.

$\boldsymbol{v} = v\boldsymbol{T}$, $\quad \dot{\boldsymbol{T}} = \mathrm{vk}\boldsymbol{N}$ [from (ii)]

From (ii), $\dot{\boldsymbol{v}} = \dot{v}\,\boldsymbol{T} + \mathrm{kv}^2\,\boldsymbol{N}$.

$$\boldsymbol{v}\times\dot{\boldsymbol{v}} = v\boldsymbol{T}\times\left(\dot{v}\,\boldsymbol{T} + \mathrm{kv}^2\,\boldsymbol{N}\right) = v\,\dot{v}\,\boldsymbol{T}x\boldsymbol{T} + v^3\,k\boldsymbol{T}\times\boldsymbol{N}$$

$$= v^3\,k\boldsymbol{T}\times\boldsymbol{N} = v^3\,k\boldsymbol{B} \quad (\text{because } \mathbf{T}x\mathbf{T} = 0)$$

Since B is a unit vector, $|\,\boldsymbol{v}\times\dot{\boldsymbol{v}}\,| = v^3\,k$, and since

$\boldsymbol{B} = \frac{\boldsymbol{v}\times\dot{\boldsymbol{v}}}{v^3\,k}$, it follows that $\boldsymbol{B} = \frac{\boldsymbol{v}\times\dot{\boldsymbol{v}}}{|\boldsymbol{v}\times\dot{\boldsymbol{v}}|}$.

(iv) Show that $\kappa = \left|\frac{\boldsymbol{v}\times\boldsymbol{c}}{v^3}\right|$

$\boldsymbol{v} = v\boldsymbol{T}$, $\quad \dot{\boldsymbol{T}} = \mathrm{vk}\boldsymbol{N}$ [from (ii)]

From (ii), $\dot{\boldsymbol{v}} = \dot{v}\,\boldsymbol{T} + \mathrm{kv}^2\,\boldsymbol{N}$.

$$|\boldsymbol{v}\times\dot{\boldsymbol{v}}\,| = |\,v\boldsymbol{T}\times\left(\dot{v}\,\boldsymbol{T} + \mathrm{kv}^2\,\boldsymbol{N}\right)|$$

$$= \left| v\,\dot{v}\,\boldsymbol{T} x \boldsymbol{T} + v^3\,k \boldsymbol{T} \times \boldsymbol{N} \right|$$

$$= \left| 0 + v^3\,k\boldsymbol{B} \right|$$

Then assuming positive curvature,

$$\kappa = \left| \frac{\boldsymbol{v}\times\dot{\boldsymbol{v}}}{v^3} \right|$$

(v) Show that $\tau = \frac{(\boldsymbol{v}\times\dot{\boldsymbol{v}})\bullet\ddot{\boldsymbol{v}}}{|\boldsymbol{v}\times\dot{\boldsymbol{v}}|^2}$

$$(\boldsymbol{v}\times\dot{\boldsymbol{v}})\bullet\ddot{\boldsymbol{v}} = v^3\,k\boldsymbol{B}\bullet\ddot{\boldsymbol{v}} \qquad (1)$$

Since any terms of $\ddot{\boldsymbol{v}}$ containing **T** or **N** produce a zero outcome in the dot product, we need only find a term with **B**.

$$\ddot{\boldsymbol{v}} = \mathrm{d}(\dot{\boldsymbol{v}})/\mathrm{dt} = \mathrm{d}(\dot{v}\,\boldsymbol{T} + \mathrm{kv}^2\,\boldsymbol{N})/\mathrm{dt}$$

$d(\dot{v}\,\boldsymbol{T})/\mathrm{dt} = \ddot{v}\,\boldsymbol{T} + \dot{v}\,\dot{\boldsymbol{T}}$, and $\dot{\boldsymbol{T}}$ points in the direction of N, so the term with **T** does not yield a term we can use.

$d(\mathrm{kv}^2\,\boldsymbol{N})/\mathrm{dt} = \left(d(\mathrm{kv}^2)/\mathrm{dt}\right)\boldsymbol{N} + \mathrm{kv}^2\,\dot{\boldsymbol{N}}$, so the term with N does not yield an useful term, but $\mathrm{kv}^2\,\dot{\boldsymbol{N}} = \mathrm{kv}^2(-\kappa\boldsymbol{T} + \tau\boldsymbol{B})\,\frac{ds}{dt}$. Since we only need the term with **B**, and because $v = \mathrm{ds/dt}$, we can write

$$\ddot{\boldsymbol{v}} \mapsto \mathrm{kv}^3\,\tau\boldsymbol{B}$$

Then, returning to (1), and substituting for $\ddot{\boldsymbol{v}}$,

$$(\boldsymbol{v}\times\dot{\boldsymbol{v}})\bullet\ddot{\boldsymbol{v}} = v^3\,k\boldsymbol{B}\bullet\mathrm{kv}^3\,\tau\boldsymbol{B} = v^6\,k^2\,\tau$$

Derive from (iv) that $|\,\boldsymbol{v}\times\dot{\boldsymbol{v}}\,| = |\,v^3\,k\,|$, so dividing both sides by $\mathrm{v}^6\ \mathrm{k}^2$,

$$\tau = \frac{(\boldsymbol{v}\times\dot{\boldsymbol{v}})\bullet\ddot{\boldsymbol{v}}}{v^6\,k^2} = \frac{(\boldsymbol{v}\times\dot{\boldsymbol{v}})\bullet\ddot{\boldsymbol{v}}}{(v^3\,k)^2} = \frac{(\boldsymbol{v}\times\dot{\boldsymbol{v}})\bullet\ddot{\boldsymbol{v}}}{|\boldsymbol{v}\times\dot{\boldsymbol{v}}|^2}$$

This is what we wanted.