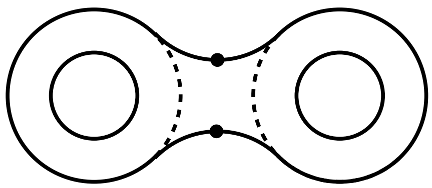
Needham, VDGAF, Chapter 20, Exercise 33, p. 227
Palmer, September 1 , 2023, 6:30 pm PST

Out[261]=



33. ***Vector Fields on $\mathcal{S}_g$ with $-\mathcal{X}$ Saddle Points***. *Show (by drawing it!) that it is possible to construct a vector field on $\mathcal{S}_g$ (with $g \geq 2$) such that there are precisely $(2g - 2) = -\mathcal{X}$ singular points, each with $\mathcal{I} = -1$. (Hints: (A) First tackle $g = 2$; the generalization to higher genus turns out to be obvious.*

Figure 1 shows a three-dimensional double torus with two saddle points and no singular points. For the double torus, $g = 2$ and $\mathcal{X} = -2$. This is not a honey-flow diagram, so we can place as many singular points as we like. Two saddle points with $\mathcal{I} = -1$ solve the equation.

$$2g - 2 = 2 \cdot 2 - 2 = -\mathcal{X} = 2 \text{ (= number of singular points)}$$

Out[835]=

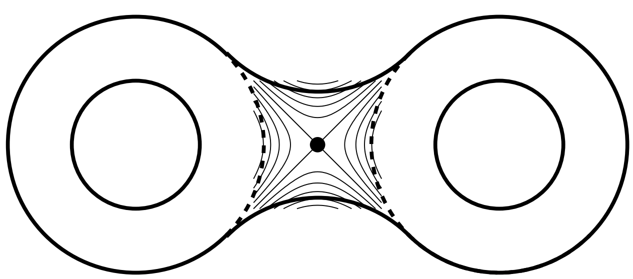


Figure 2. One of two singular points and fields.

(B) Picture $\mathcal{S}_2$ as two bagels still joined together. Next, imagine this surface crushed almost flat under a great weight, so that it resembles two joined annuluses. Finally confirm (by drawing it!) that it is possible to place one saddle point on the "top" surface, and one on the "bottom," and to have the top and bottom fields agree where they meet, at the edges.)

After squashing the 3-d image in Figure 1, move the two points from Figure 1 to become central points on the almost flat top and bottom surfaces. Only the top surface is visible in Figure 2. Since the hyperbolas are symmetrical on both x and y axes, the field lines necessarily agree where they meet along the edges.