Needham, VDGAF, Chapter 20, Exercise 32, p. 227
Palmer, August 25 , 2023, 12 noon PST

32. A Vector Field on S_g with One Singular Point? *... In this general case, is it always possible to generalize the dipole field on the sphere, to find a flow with only one singular point of index χ? [Hint: A new field can be created by coalescing two or more it its singular points into one....]*

The general case pertains to closed, connected, orientable surfaces with two or more holes, as described on page 166. Since the closed surface is connected and continuous, one can imaging moving every point to one location. But there may be constraints. If we restrict the question to the coalescence of sources and sinks, then it appears that it is always possible. But what if one attempts to merge a saddle point with a source or a sink? If one moves a saddle point to the location of a source or a sink, is it still a saddle? Perhaps that doesn't matter, because when one merges a source and a sink, the result can be regarded as something new, or as both, a source-sink. On the other hand, there is a difference in curvature between saddles on the one hand, and sinks and sources on the other. Sinks and sources both have positive curvature. Saddles have negative curvature. One cannot have both positive and negative curvature at one point. It must be one or the other. One could merge them, but in so doing, one either loses the saddle or the source/sink.

Are there other conditions which might prevent them from coalescing? One such condition might be if streamlines intersect one another. Is it a problem if the streamlines from, say, a source, intersect with the streamlines from a saddle? I will guess that this would not be a problem, because flows without singular points may cross as in [19.9], and, depending on the physical model, flows may change directions (as with the dipole on the sphere), merge (as with the dipole?), or diverge (as with the dipole) where singular points merge. Furthermore, on p. 207, we learn infinitely many, topologically distinct, regular flows can be constructed by joining the end of one streamline to the start of another to compose the fundamental group of the surface. So they could merge and join.

I conclude, based on the argument from curvature, that it is always possible to move singular points and merge them with other points to create a flow with only one singular point, but it is not always possible to do so without losing their character as sinks, sources, or saddles.