Needham, VDGAF, Chapter 20, Exercise 3, p. 219.
Palmer, December 16, 2022, 8:30 am PT

3. The Darboux Vector*. As the Frenet frame moves along its curve, it rotates. Verify by calculation that its instantaneous angular velocity vector is*

$$\boldsymbol{A} \equiv \tau\,\boldsymbol{T} + \kappa\,\boldsymbol{B}.$$

... Hint: Use (9.3) to prove that

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}' = \boldsymbol{A} \times \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}.$$

We use the hint and (9.3)

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}' = [\Omega] \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \kappa \boldsymbol{N} \\ -\kappa \boldsymbol{T} + \tau \boldsymbol{B} \\ -\tau \boldsymbol{N} \end{pmatrix}$$

Then if $\boldsymbol{A} \equiv \tau\,\boldsymbol{T} + \kappa\,\boldsymbol{B}$,

$$\boldsymbol{A} \times \boldsymbol{T} = \tau \boldsymbol{T} \times \boldsymbol{T} + \kappa \boldsymbol{B} \times \boldsymbol{T} = 0 + \kappa \mathbf{N}$$

$$\boldsymbol{A} \times \boldsymbol{N} = \tau \boldsymbol{T} \times \boldsymbol{N} + \kappa \boldsymbol{B} \times \boldsymbol{N} = \tau \boldsymbol{B} - \kappa \boldsymbol{T} \quad \text{(- sign by reverse of } \mathbf{N}\times\mathbf{B}\text{)}$$

$$\boldsymbol{A} \times \boldsymbol{B} = \tau \boldsymbol{T} \times \boldsymbol{B} + \kappa \boldsymbol{B} \times \boldsymbol{B} = -\tau \boldsymbol{N} + 0 \quad \text{(- sign by reverse of } \mathbf{B}\times\mathbf{T}\text{)}$$

Then one can write

$$\begin{pmatrix} \boldsymbol{A} \times \boldsymbol{T} \\ \boldsymbol{A} \times \boldsymbol{N} \\ \boldsymbol{A} \times \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \kappa \boldsymbol{N} \\ -\kappa \boldsymbol{T} + \tau \boldsymbol{B} \\ -\tau \boldsymbol{N} \end{pmatrix} = \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}'$$

Since $\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}$ is not a vector, one can write

$$\boldsymbol{A} \times \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} \times \boldsymbol{T} \\ \boldsymbol{A} \times \boldsymbol{N} \\ \boldsymbol{A} \times \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}'$$

which proves the hint where $\boldsymbol{A} \equiv \tau\,\boldsymbol{T} + \kappa\,\boldsymbol{B}$. Then if angular velocity of the Frenet frame is the simultaneous angular rate of change of the three Frenet axes with respect to a tangent to the curve at time t, we have also verified $\boldsymbol{A} \equiv \tau \boldsymbol{T} + \kappa \boldsymbol{B}$.