Needham, VDGAF, Chapter 20, Exercise 27, p. 227
Palmer, August 5 , 2023, 4 pm PST

27. Hopf's Proof That $\chi(S_g) = 2 - 2g$. *In the text, we proved $\chi(S_g) = 2 - 2g$ by first showing that $\chi(\mathbb{S}^2) = 2$, and then showing that gluing on a "Toblerone ® handle" (see [18.9]) reduces χ by 2 (see [18.10]). Here is an elegant alternative argument, due to Hopf (1956, pp. 8-10).*

(i) Glue together two Toblerone ® handles to create a topological torus, S_1, and deduce that $\chi(S_1) = 0$.

$V = 0$, $E = 3$, $F = 3$

From (18.1), $\chi(S_1) = 0 - 3 + 3 = 0$

(ii) Remove one triangle from this torus, and remove one triangle from a general surface S_g of genus g. Now glue the edges of the two triangular holes to each other, thereby creating a new closed surface S_{g+1} of one higher genus. Deduce that $\chi(S_{g+1}) = \chi(S_g) - 2$.

One could deduce this as a hypothesis from [16.2], but using (18.7), $\chi(S_g) = 2 - 2g$. Then

$$\chi(S_{g+1}) = 2 - 2(g + 1) = -2g = (2 - 2g) - 2 = \chi(S_g) - 2.$$

Done.

I tried another approach, making the V, E, and F of the Euler characteristic explicit. The removing and gluing process adds 6 new edges to S_g (three at the join, three from the Toblerone® torus circular edges). It also adds 3 vertices at the join and 3 faces from the Toblerone® torus. Then

$$\chi(S_{g+1}) = \chi(S_g) + 3 - 6 + 3 = \chi(S_g)$$

This contradicts the given answer. If we assume that the Toblerone® torus loses the three long edges of the Toblerone® and reduces the faces to 1, then we have (added to or subtracted from $\chi(S_g)$): V = 3, E = 3, F = 1.

$$\chi(S_{g+1}) = \chi(S_g) + 3 - 3 + 1 = \chi(S_g) + 1$$

This also contradicts the given answer.

(iii) Use (i) and (ii) to deduce that $\chi(\mathbb{S}^2) = 2$.

In (ii) we established that $\chi(S_{g+1}) = \chi(S_g) - 2$ and $\chi(S_1) = 0$. The first equation can be rewritten

$$\chi(S_g) = \chi(S_{g+1}) + 2.$$

$\mathbb{S}^2$ is just the unit sphere. Spheres have genus 0. Then

$$\chi(\mathbb{S}^2) = \chi(S_0) = \chi(S_1) + 2 = 0 + 2 = 2$$