Needham, VDGAF, Chapter 20, Exercise 25, p. 226
Palmer, July 25 , 2023, 10:40 pm PST

25. There Are Only Five Topologically Regular Polyhedra. *Let us use the same (n,m) notation as in the previous exercise, only now the faces of our topological polyhedron can be curved, irregular, and/or with wavy edges. as usual, let V, E, and F denote the number of vertices, edges, and faces.*

From Dynamic Errata:
226 Ex. 25 Swap V and E in parts (i) and (ii). Petra Axolotl

(i) Explain why $E = \mathrm{Fn}/2$. (corrected)

Fn gives the total number if the edges of every n-gon are counted separately and then the edges for each are multiplied by the number of n-gons (faces). In the polyhedron, every edge is shared between two faces, so the net number of edges is obtained by dividing Fn by two.

(ii) Explain why $V = \mathrm{Fn}/m$. (corrected)

Fn also gives the total number of vertices if the vertices of every n-gon are multiplied by the number of n-gons (faces). But that number must be divided by the number of n-gons (faces) that conjoin at each vertex. That number is m.

(iii) By substituting the previous two equations into Euler's Polyhedral formula, deduce that

$$F = \frac{4m}{2(m+n)-mn}$$

Rewrite Euler's formula $V + F = E + 2$:

$$\mathrm{Fn}/m + F = \mathrm{Fn}/2 + 2$$
$$\mathrm{Fn}/m - \mathrm{Fn}/2 + F = 2$$
$$F(n/m - n/2 + 1) = 2$$
$$\mathrm{F2m}\,(n/m - n/2 + 1) = 4m \quad \text{(multiply both sides by 2m)}$$

$$F = \frac{4m}{(2n - \mathrm{mn} + 2m)} = \frac{4m}{2(m+)\,n - \mathrm{mn}}$$

This is what we wanted.

(iv) Deduce that $\Omega\,(n,\ m) = 2\,(m + n) - mn > 0$.

Applying $\Omega\,(n,\,m) < 0$ to the equation for F in part (iii), there would be a negative number of faces, which is a contradiction, or undefined, or contrary to the construction. If $\Omega\,(n,\,m) = 0$, then F is a singularity, which is contrary to the assertion that *There Are Only Five Regular Polyhedra. Hence,* Ω(n,m) > 0.

(v) Think of (m, n) as grid points in $\mathbb{R}^2$*. In the relevant region* ($m \geq 3$ *and* $n \geq 3$) *mark each grid point with the value* $\Omega(m, n)$*, noting and using the symmetry about about the line* $n = m$*.*

The greatest number of sides in one n-gon of a topological polyhedron is 5 in the dodecahedron. The greatest number of sides forming a vertex of a topological polyhedron is 5 in the icosahedron. Since both m and n are greater than or equal to 3, we obtain Table 1.

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & -5 \end{pmatrix}$$

Table 1. Ω(*n*, *m*)
where m = 3, 4, 5 by row
n = 3, 4, 5 by column

The table is symmetrical about the line *n* = *m*. It is not clear how one should use this information except to exclude the 0 and negative values, leaving only the first row and first column as valid topological polyhedrons. If we place the name prefixes of the Platonic solids in the (m,n) grid, we get

$$\begin{pmatrix} \text{"Tetra"} & \text{"Cube"} & \text{"Dodec"} \\ \text{"Octa"} & \text{"---"} & \text{"---"} \\ \text{"Icosa"} & \text{"---"} & \text{"---"} \end{pmatrix}$$

Table 2. Names of Platonic solids by m row and n column.

The three in in row 1 have vertices that conjoin three sides. The three in column 1 have faces that are 3-gons. All five of the Platonic solids have one or the other characteristic. This seems barely more significant than numerology.

(vi) Deduce that the only solutions (n, m) are the same ones we found in the previous exercise: the five Platonic solids, but now topologically deformed.

By the Omega criterion, the only valid topological polyhedra are those in the first row or the first column of Table 1. The Platonic solids in Table 2 have the same pattern of (m, n) and Omega values. Therefore, they have the same (m, n) solutions.

None of the Euler characteristics---V, F, E---or (m, n) depend upon curvature. They depend only on (1) the existence of points (vertices) formed by the intersection of three or more smooth non-self-intersecting curves (edges) or surfaces (faces); (2) smooth, non-self-intersecting curves formed by conjoined surfaces; and (3) smooth non-intersecting surfaces themselves, where these form a single closed construction of points, curves and surfaces. The lines and surfaces may have positive or negative curvature, or both. The regular polyhedra (Platonic solids) are special cases of the topologically regular polyhedra. Hence the solutions (n, m) are the same, and there are only five topologically regular polyhedra.