Needham, VDGAF, Chapter 20, Exercise 23, p. 225
Palmer, August 1 , 2023, 3:20 pm PST

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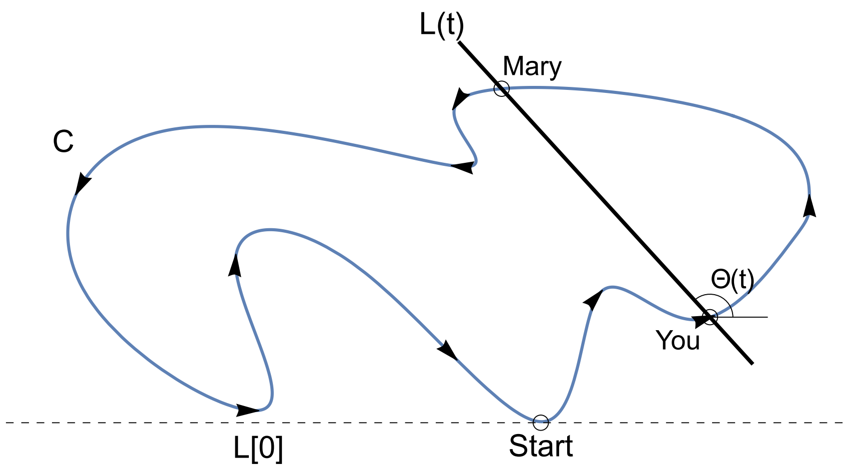

Figure 1. Facsimile of figure p. 225

23. Hopf's Umlaufsatz*. ... Imagine that the (smooth and "simple") closed curve C of the theorem is convoluted hiking/running path snaking through sand dunes... The entrance is at the southernmost point ...; the rest of C lies to the north .... As illustrated, let* $\Theta(t)$ *be the angle of the line L(t) connecting you ... and your friend, Mary ..., at time t. Initially, ..., L(0) ... is horizontal* [dashed line], *with* $\Theta(0) = 0$.

(i) Mary decides to do a warm-up lap, but you stay at the start and just watch her run. As she runs the length of C, counterclockwise, you turn your head to follow. Explain why your line of sight must execute a net rotation of π *by the time Mary returns to you.*

I see Mary at angle of 0 radians as she departs and I stay in place, and see her at π radians as she returns. It doesn't matter how her route winds or whether the direction of my gaze sometimes exceeds π. The direction of my gaze along L undergoes a net rotation of π, so L must undergo the same net rotation.

(ii) The reason is that L is caused to rotate by π by Mary's traversal, and again by π by your traversal.

(iii) You race, but Mary always wins. I see this is no different from (ii).

(iv) There is no question. The construction is the same, except that you are close enough to say that L(t) has become the tangent line. The fact that the rotation occurs as the tangent traverses the circle shows that the velocity has executed a full revolution and effects the proof of Hopf's theorem.