Needham, VDGAF, Chapter 20, Exercise 22, p. 225
Palmer, July 20 , 2023, 7:45 pm PST

22. Holy Broken Gauss-Bonnet, Batman! *Recall our definition of a wormhold [16.7], and also recall that it as total curvature* $\mathcal{K} = -4\pi$.

(i) If n small bagels are placed at the vertices of a large regular n-gon, and are joined together with wormhole connectors along the edges of the n-gon into a surface $\mathcal{S}$*, explain why* $\mathcal{K}(\mathcal{S}) = -4\pi n$*. Why is this not a violation of GCB, as it appears to be.*

On page 172, Needham calculated the curvature of bagels (tori) joined sequentially by wormhole bridges as

$$\mathcal{K}(\mathcal{S}_g) = g \cdot \mathcal{K}(\text{bagel}) + (g-1) \cdot \mathcal{K}(\text{bridge}) = g \cdot 0 + (g-1) \cdot (-4\pi) = 4\pi(1-g).$$

The bagels have $\mathcal{K}$ of zero and the bridges have value -4π. The number of bridges $(g-1)$ is one less than the number of bagels. The n-gon requires a different calculation, because the number of bridges is n, which happens to be the same as the number of bagels, so the calculation is:

$$\mathcal{K}(\mathcal{S}_g) = g \cdot \mathcal{K}(\text{bagel}) + n \cdot \mathcal{K}(\text{bridge}) = g \cdot 0 + n \cdot (-4\pi) = -4\pi n$$

“in accord with GGB.”

(ii) Suppose n = 4, so that the bagels are at the corners of a square, joined along the four edges. Now add a fifth wormhole connector along one of the diagonals, connecting two previously disconnected bagels. What is the total curvature now, and why is this not a violation of GGB, as it appears to be.

$$\mathcal{K}(\mathcal{S}_g) = g \cdot \mathcal{K}(\text{bagel}) + n \cdot \mathcal{K}(\text{bridge}) = g \cdot 0 + 5 \cdot (-4\pi) = -20\pi$$

This is not a violation of GGB because it follows the GGB principle of adding zero curvatures of the bagels to the negative curvatures of the wormhole connectors. Compare the torus of genus 2 (p. 167). It has two bagels of zero $\mathcal{K}$ and one connection where $\mathcal{K} = -4\pi$. The total curvature of the double torus is $2\pi\mathcal{X} = 2\pi(-2) = -4\pi$.