Needham, VDGAF, Chapter 20, Exercise 21, p. 224
Palmer, July 17 , 2023, 1:20 pm PST

21. Genus-Dependent Predictions of GGB.

(i) Use GGB to deduce that any topological sphere must have regions of positive curvature.

The genus of a topological sphere is 0 According to the GGB, the total curvature of the sphere is

$$\mathcal{K}(\mathcal{S}_g) = 4\,\pi(1-0) = 4\,\pi.$$

If the topological is a canonical sphere, it can be regarded as a single region of positive curvature, and any and all subregions would also have positive curvature. If the topological sphere has regions of negative curvature, it must have one or more regions of positive curvature that surpass the negative curvature by 4π.

(ii) Use GGB to deduce that any topological donut must have regions of positive curvature.

By the GGB,

$$\mathcal{K}(\mathcal{S}_g) = 4\,\pi(1-1) = 0$$

Since the donut clearly has a region of negative curvature surrounding the donut hole, there must be one or more regions of positive curvature that permit the total curvature to sum to zero.

Out[515]=

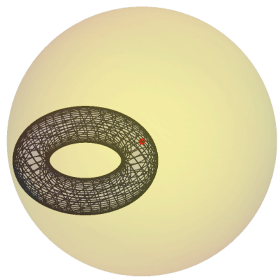

Figure 1. Sphere shrinks to touch genus 1 torus

Out[535]=

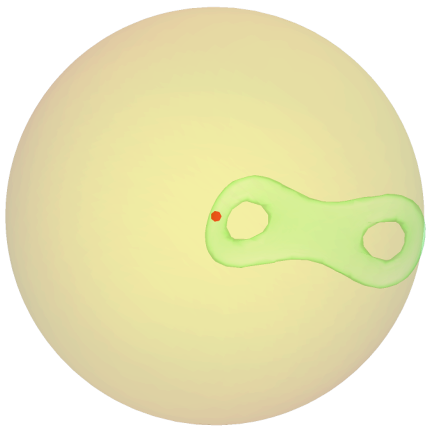

Figure 2. Sphere shrinks
to touch genus 2 torus
It's not easy being green.

(iii) In fact, every smooth closed surface $\mathcal{S}_g$ *with* $g \geqslant 2$ *must also have a point of positive curvature, though GGB alone no longer guarantees this. Let* $S(r)$ *be a sphere of radius r centred at an arbitrary point inside* $\mathcal{S}_g$*. Imagine we start with a sufficiently large value of r that S(r) contains* $\mathcal{S}_g$*. Now shrink the sphere until it makes first contact with* $\mathcal{S}_g$*, where r = R, say. Argue that at the point of contact with* $S(r)$*,* $\mathcal{K} > \left(1/R^2\right) > 0$*. Hint: Consider the principal directions at the point of contact.*

From my perspective, there are two points of ambiguity in the question. First, what is meant by "inside $\mathcal{S}_g$"? My first choice would be a point in the space inside the 3-dimensional object created by the surface. This choice is illustrated with the red dots in figures 1 and 2. My second choice would be in the surface itself. My third choice would be somewhere in the 3-dimensional region created if one wrapped the surface with something that would obscure it from view. The wrapping would not have to touch the negative surfaces. Second, I don't follow what is meant by "*first contact with $\mathcal{S}_g$, where $r = R$*", as I am unsure of what R stands for. I will take R to be the radius of curvature of the outside of a ring of the torus at the point of contact, where $r = R$ asymptotically. R is **not** the curvature of a ring defining a transverse cut through a ring of the torus.

These qualms aside, it seems clear by inspection that since the surface of $\mathcal{S}_g$ is smooth, shrinking S(r) must first touch a hill on $\mathcal{S}_g$. The hill has two principle directions. If one plots two curves on $\mathcal{S}_g$ from a point of contact p on one ring (donut) of an n-torus to points lying in the principle directions, both curves bend away from the tangent plane at p and towards a centre of curvature. Let κ_1 be the curvature on R, and let κ_2 be the curvature in the orthogonal direction of the transverse cut through the torus. Hence, both have positive curvature. The curvature κ_2 is obviously greater than κ_1, so $R_1 > R_2$. Then, letting $R = R_1$ be the local 2-dimensional radius of curvature at points on the principal direction of curvature infinitesimally close to the point of contact, $\mathcal{K} = \kappa_1 \kappa_2 = 1/R_1 R_2 > 1/R^2 > 0$.

Note: For plotting 3D regions in Mathematica, see <https://mathematica.stackexchange.com/questions/237592/creating-a-plot-for-a-solid-torus>.